WEIERSTRASS' THEOREM AND KNESER'S THEOREM ON TRANSVERSALS FOR THE MOST GENERAL CASE OF AN EXTREMUM OF A SIMPLE DEFINITE INTEGRAL*

BY

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Introduction.

Let

$$y = Y(x, a) \tag{1}$$

denote a one-parameter set of extremals for the integral

$$J = \int_{x_0}^{x_1} F(x, y, y')\, dx,$$

passing *through a fixed point* (ξ, η) and furnishing a field about an arc $\mathfrak{E}_0$ of the particular extremal $a = a_0$, and let

$$a = \mathfrak{a}(x, y)$$

denote the inverse function of the field, obtained by solving (1) with respect to a.

Further, let the function $W(x, y)$ be defined by the two equations

$$U(x, a) = \int_{\xi}^{x} f(x, Y, Y')\, dx,$$

$$W(x, y) = U(x, \mathfrak{a}),$$

so that the function $W(x, y)$—which we will call the *field integral*—represents the value of the integral J taken from the fixed point (ξ, η) to an arbitrary point (x, y) of the field along the unique extremal of the field passing through the point (x, y).

Then the following two fundamental formulas hold for the partial derivatives of the field integral: †

$$\begin{aligned} \frac{\partial W}{\partial x} &= f(x, y, p) - p f_{y'}(x, y, p), \\ \frac{\partial W}{\partial y} &= f_{y'}(x, y, p), \end{aligned} \tag{2}$$

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† See, for instance, BOLZA, *Lectures on the Calculus of Variations*, p. 266.

where $p = p(x, y)$ is the slope of the extremal of the field passing through (x, y), at that point:

$$p(x, y) = Y'(x, a).$$

We shall call these formulas *Hamilton's Formulas*, because they have first been given by HAMILTON* as early as 1835, even for a much more general case.

From HAMILTON's formulas follow immediately: on the one hand "*Weierstrass' theorem*"†

$$\Delta J = \int_{x_0}^{x_1} E dx,$$

by means of "WEIERSTRASS' construction"; on the other hand "*Beltrami-Hilbert's independence theorem*,"‡ which states that the expression

$$[f(x, y, p) - pf_{y'}(x, y, p)]\,dx + f_{y'}(x, y, p)\,dy \tag{3}$$

is a complete differential if the slope-function $p(x, y)$ is substituted for p.

When one attempts to extend HAMILTON's formulas and the allied theory from the particular case of a set of extremals through a fixed point to the case of *any set of extremals*, one meets at once with a difficulty. It is, in this case, not a priori clear from what point as lower limit the integral W should be taken on the different extremals of the set. This difficulty has been removed by KNESER as follows:§ He first takes the integral $W(x, y)$ from an arbitrary but fixed curve $\mathfrak{T}$ drawn across the field and computes the partial derivatives of W; it turns out that they differ from the simple expressions (2) only by an additional term in the expression of $\partial W/\partial y$. And then he so determines the curve $\mathfrak{T}$ that this additional term vanishes. He finds as solution of this problem — which we shall call "*Kneser's Problem*" — the result that the curve $\mathfrak{T}$ must be a "*transversal*" to the set of extremals forming the field. Closely connected with this result is "*Kneser's Theorem on Transversals.*"‖

HAMILTON's formulas being thus extended to any set of extremals, the corresponding extension of WEIERSTRASS' theorem on the one hand, and of BELTRAMI-HILBERT's independence theorem on the other hand follow easily.

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*Philosophical Transactions, 1835, part I, p. 77.

†WEIERSTRASS, *Berlin Lectures*, 1879; see, for instance, OSGOOD, Annals of Mathematics, ser. 2, vol. 2 (1901), p. 115, and BOLZA, *Lectures*, pp. 87, 266.

‡The theorem was first given, independently of HAMILTON's formulas, by BELTRAMI in a paper on geodesics, *Sulla teorica delle linee geodetiche*, published in 1868 (Rendiconti del Reale Istituto Lombardo, ser. 2, vol. 1, pp. 708–718, also *Opere*, vol. 1, pp. 366–373). This important paper seems to have remained unnoticed until quite recently (my own attention has been called to it by Professor KNESER). The theorem was rediscovered thirty years later by HILBERT who made it the basis of his well-known proof of WEIERSTRASS' theorem (see Göttinger Nachrichten, 1900, pp. 253–297, and Archiv für Mathematik und Physik, ser. 3, vol. 1 (1901), p. 231.)

§See KNESER, *Lehrbuch der Variationsrechnung*, § 15, and OSGOOD, l. c., p. 119.

‖See KNESER, *Lehrbuch*, p. 48, and BOLZA, *Lectures*, § 33.

A second method for the extension of WEIERSTRASS' theorem to any set of extremals has been given by HILBERT.* Instead of starting from the integral W, he starts from the differential expression (3) with an arbitrary function p of x and y, and proposes so to determine the function p that the expression (3) becomes a complete differential. He finds as solution of this problem — which we shall call (according to A. MAYER) "*Hilbert's Problem*" — the slope-function

$$p = Y'(x, \mathfrak{a}),$$

derived from *any* set of extremals, and bases on this result a new proof of WEIERSTRASS' theorem by means of his *Invariant Integral*.

HILBERT's theory has recently been extended by A. MAYER† to the so-called most general case of an extremum of a simple definite integral, in which it is required to minimize an integral of the form

$$J = \int_{x_0}^{x_1} f(x, y_1, \cdots, y_n; y_1', \cdots, y_n')\, dx, \tag{4}$$

involving n unknown functions $y_1, \cdots, y_n$ of x and their first derivatives $y_1', \cdots, y_n'$, connected by $r < n$ differential equations

$$f_\rho(x, y_1, \cdots, y_n; y_1', \cdots, y_n') = 0 \qquad (\rho = 1, 2, \cdots, r) \tag{5}$$

MAYER finds the remarkable result that for $n > 1$ not all, but only a certain class of n-parametric sets of extremals, furnish solutions of HILBERT's problem, and derives from his results a proof of WEIERSTRASS' theorem by means of HILBERT's invariant integral.

In the present paper, which has developed out of the study of Professor MAYER's memoirs, *I propose to give an analogous extension, to the same general case, of Weierstrass' original theory and of Kneser's theory* as sketched above.

I have endeavored to reach in the discussion of the general case the same degree of rigor which has been attained for the simplest type of problems. For this purpose, I make use of certain existence-theorems concerning implicit functions and systems of differential equations, which extend the results, usually given for the vicinity of *a point*, to the vicinity of *a point set;* these auxiliary theorems are given in §§ 1 and 2.

*See the references to HILBERT's papers on p. 460, third footnote; compare also OSGOOD, loc. cit., p. 121, and BOLZA, *Lectures*, § 21.

† *Ueber den Hilbert'schen Unabhängigkeitssatz in der Theorie des Maximums und Minimums der einfachen Integrale*, Leipziger Berichte, 1903, pp. 131-145; 1905, pp. 49-67 and 1905, pp. 313-314. We shall refer to these papers as "MAYER, I, II, III."

HILBERT himself has given a geometrical proof of the generalized independence theorem for the case $r = 0$, in which he reduces the case n to the case $n - 1$; see his paper, *Zur Variationsrechnung*, Göttinger Nachrichten, February, 1905.

In § 3 follows a short sketch of the reduction of EULER-LAGRANGE's differential equations to a canonical system, in §§ 4 and 5 the proof of WEIERSTRASS' theorem for a set of extremals through a fixed point, based upon a combination of HAMILTON's formulas and WEIERSTRASS' construction.

In §§ 6 and 7 the extension of KNESER's theory is given, and finally § 8 contains the application to WEIERSTRASS' theorem for MAYER's class of n-parametric sets of extremals.

§ 1. *Auxiliary theorems concerning implicit functions.*

For brevity I shall say that a function $f(x_1, \cdots, x_n)$, which is defined in* a region† $\mathbf{A}$, is of class $C^{(p)}$ in $\mathbf{A}$, when the function f itself and its partial derivatives up to the pth order (inclusive) exist and are continuous in* $\mathbf{A}$ (if necessary, after a proper extension‡ of the definition of the function $f(x_1, \cdots, x_n)$ beyond the region $\mathbf{A}$). Further I shall say that a point $(x) = (x_1, \cdots, x_n)$ lies in "*the vicinity* (ρ) *of a point set* $\mathbf{M}$" defined in the $(x_1, \cdots, x_n)$ space, if it lies in the vicinity (ρ) of at least one point of $\mathbf{M}$, i. e., if there exists a point $(\bar{x}_1, \cdots, \bar{x}_n)$ of $\mathbf{M}$ such that

$$|x_1 - \bar{x}_1| < \rho, \cdots, |x_n - \bar{x}_n| < \rho.$$

The vicinity (ρ) of the set $\mathbf{M}$ thus defined will be denoted by $(\rho)_{\mathbf{M}}$.

The following theorem holds:

LEMMA I. *Let the functions* $f_i(x_1, \cdots, x_n)$ *be of class* C' *in a region* $\mathbf{A}$*; let further* $\mathbf{C}$ *be a bounded*§ *closed*‖ *point set in the interior*¶ *of* $\mathbf{A}$ *which has the property that any two distinct points* (x'), (x'') *of* $\mathbf{C}$ *are transformed into two distinct points* (u'), (u'') *by the transformation*

$$u_i = f_i(x_1, \cdots, x_n) \qquad (i = 1, 2, \cdots, n). \tag{6}$$

If then the Jacobian

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$$

* *in* signifies *at all points of.*

† I use the word "region" for a point set which contains "inner" points, in accordance with BLISS, Annals of Mathematics, ser. 2, vol. 6 (1905), p. 49.

‡ This additional clause is necessary if we wish to cover the case where the region $\mathbf{A}$ contains also "boundary points," because the derivative of a function, as usually defined, has a meaning only for inner points of the region in which the function is defined.

§ "Borné" (JORDAN).

‖ "Abgeschlossen" (CANTOR).

¶ I. e., every point of $\mathbf{C}$ is an "inner" point of $\mathbf{A}$.

is different from zero in * $\mathbf{C}$, *a vicinity* $(\rho)_{\mathbf{C}}$ *contained in the interior of* $\mathbf{A}$ *can be assigned such that*

1) *the transformation* (6) *defines a one-to-one correspondence between the point set* $(\rho)_{\mathbf{C}}$ *and its image* $\mathbf{S}_\rho$ *in the* (u)*-space;*

2) *the set* $\mathbf{S}_\rho$ *is made up exclusively of inner points;*

3) *the inverse functions*

$$x_k = \psi_k(u_1, \cdots, u_n)$$

thus defined for $\mathbf{S}_\rho$ *are of class* C' *in* $\mathbf{S}_\rho$.

The proof of the theorem is based upon a slight modification of the method by which I have proved a special case of the theorem in § 34 of my *Lectures on the Calculus of Variations;* for the details I refer to my note, *Ein Satz über eindeutige Abbildung und seine Anwendung in der Variationsrechnung*, in one of the forthcoming numbers of the Mathematische Annalen.

If we denote by $\mathbf{E}$ the image of $\mathbf{C}$ in the (u)-space, it follows that we can assign a second positive quantity σ such that $(\sigma)_{\mathbf{E}}$ is contained in $\mathbf{S}_\rho$. Hence we have the

COROLLARY: *For every point* (u) *in* $(\sigma)_{\mathbf{E}}$ *the equations* (6) *have one and but one solution* (x) *in* $(\rho)_{\mathbf{C}}$.

This remark leads to the following

LEMMA II.† *If the functions* $f_i(x_1, \cdots, x_m; y_1, \cdots, y_n)$ *are of class* C' *in a region* $\mathbf{A}$ *in the* (x, y)*-space and if the* n *equations*

$$f_i(x_1, \cdots, x_n; y_1, \cdots, y_n) = 0 \qquad (i = 1, 2, \cdots, n), \tag{7}$$

are fulfilled for all the points of a point set $\mathbf{C}$ *of the following properties:*

1) *the set* $\mathbf{C}$ *is bounded and closed, and lies in the interior of* $\mathbf{A}$*;*

2) *if* (x', y'), *and* (x'', y'') *are two distinct points of* $\mathbf{C}$, *then always* $(x') \neq (x'')$*;*

3) *the Jacobian*

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}$$

is different from zero in $\mathbf{C}$*;*

then, if $\mathbf{X}$ *denote the "projection"* ‡ *of* $\mathbf{C}$ *into the* (x)*-space, two positive quantities* ρ *(first) and* σ *(second) can be assigned such that*

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* I. e., at every point of $\mathbf{C}$.

† This is an extension of the ordinary theorem on implicit functions which refers to the solution of a system of equations in *the vicinity of a point*, see GENOCCHI–PEANO, *Differentialrechnung und Grundzüge der Integralrechnung*, nos. 110–117; JORDAN, *Cours d'Analyse*, vol. 1, nos. 91, 92; OSGOOD, *Lehrbuch der Functionentheorie*, vol. 1, pp. 47–55.

‡ The projection of the point $(x_1', \cdots, x_m', y_1', \cdots, y_n')$ into the (x)-space is the point $(x_1', \cdots, x_m')$.

1) *for every* (x) *in* $(\sigma)_{\mathbf{X}}$ *there exists one and but one solution* (y) *of the equations* (7) *for which* (x, y) *lies in* $(\rho)_{\mathbf{C}}$;

2) *the corresponding implicit functions*

$$y_i = \phi_i(x_1, \cdots, x_n)$$

are of class C' *in* $(\sigma)_{\mathbf{X}}$.

To prove lemma II apply lemma I to the system of equations

$$u_h = x_h, \qquad u_{m+i} = f_i(x_1, \cdots, x_m; y_1, \cdots, y_n) \quad (h = 1, 2, \cdots, m;\ i = 1, 2, \cdots, n),$$

and put afterwards $u_{m+i} = 0$.

§ 2. *An existence theorem for systems of ordinary differential equations.*

We consider a system of n differential equations

$$\frac{dx_k}{dt} = f_k(t, x_1, \cdots, x_n) \qquad (k = 1, 2, \cdots, n), \tag{8}$$

in which the functions f_k are continuous in a region **A** and admit continuous first partial derivatives with respect to $x_1, \cdots, x_n$ in the interior of **A**.

Then if $A(\tau, \xi_1, \cdots, \xi_n)$ is a point in the interior of the region **A**, there exists, according to CAUCHY's existence theorem, a unique solution of (8) which passes through the point A and which is of class C' in the vicinity of $t = \tau$. We denote * it by

$$x_k = \phi_k(t;\ \tau, \xi_1, \cdots, \xi_n),$$

so that

$$\phi_k(\tau;\ \tau, \xi_1, \cdots, \xi_n) = \xi_k.$$

Under these circumstances the following theorem holds:

LEMMA III. *Let*

$$x_k = x_k^0(t), \qquad t_0 \leqq t \leqq t_1 \tag{9}$$

be a solution of (8) *which lies* IN THE INTERIOR OF THE REGION **A**; *let* τ_0 *be a particular value of* t *in the interval* $(t_0 t_1)$ *and put*

$$\xi_k^0 = x_k^0(\tau_0).$$

Then three positive quantities d, e_0, e_1 *can be assigned such that for every point* $A(\tau, \xi_1, \cdots, \xi_n)$ *for which*

$$|\tau - \tau_0| \leqq d, \qquad |\xi_k - \xi_k^0| \leqq d \qquad (k = 1, 2, \cdots, n),$$

the solution

$$x_k = \phi_k(t;\ \tau, \xi_1, \cdots, \xi_n) \tag{10}$$

* I adopt the notation of BLISS in his article, *The solutions of differential equations of the first order as functions of their initial values*, Annals of Mathematics, ser. 2, vol. 6 (1905), pp. 49–68. To the same article the reader is referred for the literature of the subject.

exists, is of class C' and lies in the interior of **A** FOR EVERY t IN THE INTERVAL

$$t_0 - e_0 \leqq t \leqq t_1 + e_1.$$

For $\tau = \tau_0, \xi_1 = \xi_1^0, \cdots, \xi_n = \xi_n^0$, *the solution* (10) *coincides with the solution* (9):

$$\phi_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_k^0) \equiv x_k^0(t), \qquad \textit{in } (t_0 t_1).$$

For analytic functions f_k the theorem has been given, in a slightly different form, by KNESER in his *Lehrbuch der Variationsrechnung* (§ 27), for non-analytic functions by A. C. LUNN in his Thesis (Chicago, 1904), as yet unpublished. On account of the importance of the theorem for the Calculus of Variations, I add here still another proof:

Since the solution (10) is supposed to lie in the interior of the region **A**, it can be "continued" * beyond the interval $(t_0 t_1)$ to a slightly larger interval $(t_0 - e_0, t_1 + e_1)$ without leaving the interior of **A**, and without ceasing to be of class C'. From theorems on continuous functions it follows then that we can assign a positive quantity σ such that the domain $[\sigma]$ defined by the inequalities

$$[\sigma]: \qquad t_0 - e_0 \leqq t \leqq t_1 + e_1, \qquad |x_k - x_k^0(t)| \leqq \sigma \quad (k = 1, 2, \cdots, n),$$

still lies in the interior of **A**. In this domain $[\sigma]$ the functions f_k are continuous and satisfy LIPSCHITZ'S condition †

$$(11)\quad |f_k(t, x_1', \cdots, x_n') - f_k(t, x_1'', \cdots, x_n'')| < K \sum_{i=1}^{n} |x_i' - x_i''| \quad (k = 1, 2, \cdots, n),$$

with a certain value of the constant K.

We now coördinate with every point $(t, x_1, \cdots, x_n)$ of the domain

$$\tilde{\mathbf{A}}: \qquad t_0 - e_0 \leqq t \leqq t_1 + e_1, \qquad -\infty < x_k < +\infty \quad (k = 1, 2, \cdots, n),$$

a point $(t, \tilde{x}_1, \cdots, \tilde{x}_n)$ of $[\sigma]$ by means of the following definition:

$$\tilde{x}_i = \begin{cases} x_i & \text{when} \quad x_i^0(t) - \sigma \leqq x_i \leqq x_i^0(t) + \sigma, \\ x_i^0(t) + \sigma & \text{when} \quad x_i > x_i^0(t) + \sigma, \\ x_i^0(t) - \sigma & \text{when} \quad x_i < x_i^0(t) - \sigma, \end{cases}$$

and define ‡ the functions $\tilde{f}_k(t, x_1, \cdots, x_n)$ for the domain $\tilde{\mathbf{A}}$ by the equation

$$\tilde{f}_k(t, x_1, \cdots, x_n) = f_k(t, \tilde{x}_1, \cdots, \tilde{x}_n).$$

* Compare BLISS, loc. cit., p. 52.

† Compare BLISS, loc. cit., p. 50, footnote, and p. 60, footnote.

‡ The definition of $\tilde{f}_k$ admits of a simple geometrical interpretation in the case $n = 1$, if the value of $\tilde{f}_k$ is represented by a third coördinate perpendicular to the (t, x)-plane.

From this definition it follows easily that the functions $\tilde{f}_k$ are continuous in the domain $\tilde{\mathbf{A}}$; moreover there exists a positive quantity G such that

$$|\tilde{f}_k(t, x_1, \cdots, x_n)| \leqq G \qquad \text{in } \tilde{\mathbf{A}};$$

and finally the functions $\tilde{f}_k$ satisfy LIPSCHITZ's condition in $\tilde{\mathbf{A}}$ with the same value of the constant K which occurs in (11).

Hence follows, according to CAUCHY's existence theorem: If

$$t_0 - e_0 \leqq \tau \leqq t_1 + e_1$$

and if $\xi_1, \cdots, \xi_n$ is *any* system of values, the system of differential equation,

$$\frac{dx_k}{dt} = \tilde{f}_k(t, x_1, \cdots, x_n) \qquad (k=1, 2, \cdots, n), \tag{12}$$

has a unique solution passing through the point $A(\tau, \xi_1, \cdots, \xi_n)$:

$$x_k = \tilde{\phi}_k(t;\ \tau, \xi_1, \cdots, \xi_n),$$

and this solution exists and is of class C' *throughout the whole* interval* $(t_0 - e_0, t_1 + e_1)$.

Moreover it follows from PEANO's inequality † that

$$|\tilde{\phi}_k(t;\ \tau, \xi_1, \cdots, \xi_n) - \tilde{\phi}_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_n^0)| < e^{nK|t-\tau|}\left\{ G|\tau - \tau_0| + \sum_{i=1}^{n} |\xi_i - \xi_i^0| \right\}. \tag{13}$$

On the other hand it follows from the definition of the functions $\tilde{f}_k$ that every solution of (12), which lies wholly in $[\sigma]$, satisfies at the same time (8), and vice versâ. Hence we infer in the first place that the solution

$$x_k = \phi_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_n^0)$$

of (8) — which coincides with the solution (9) since both pass through the point $(\tau_0, \xi_1^0, \cdots, \xi_n^0)$ — satisfies also (12); therefore we have

$$\tilde{\phi}_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_n^0) \equiv \phi_k(t;\ \tau_0, \xi_1^0, \cdots, \xi_n^0) \equiv x_k^0(t).$$

It follows now from (13) that if we take the positive quantity d sufficiently small and choose τ and ξ_k so that

$$|\tau - \tau_0| \leqq d, \qquad |\xi_k - \xi_k^0| \leqq d \qquad (k=1, 2, \cdots, n), \tag{14}$$

the solution

$$x_k = \tilde{\phi}_k(t;\ \tau, \xi_1, \cdots, \xi_n) \qquad (t_0 - e_0 \leqq t \leqq t_1 + e_1),$$

of (12) lies in $[\sigma]$ and satisfies therefore at the same time (8), and since it satis-

* Compare PICARD, *Traité d'Analyse*, vol. 3 (ed. 1896), p. 91.

† Compare BLISS, loc. cit., pp. 55 and 62.

fies the same initial conditions as (10), the two solutions must be identical; hence also the solution (10) exists, is of class C' and lies in the interior of **A** for every t in the interval $(t_0 - e_0, t_1 + e_1)$, if the condition (14) is satisfied, Q.E.D.

From the general existence theorems* for differential equations it follows further that the functions

$$\phi_k(t;\ \tau,\ \xi_1,\ \cdots,\ \xi_n)$$

considered as functions of their $n+2$ arguments are continuous together with the derivatives

$$\frac{\partial \phi_k}{\partial t},\qquad \frac{\partial \phi_k}{\partial \tau},\qquad \frac{\partial \phi_k}{\partial \xi_i},\qquad \frac{\partial^2 \phi_k}{\partial t\,\partial \tau},\qquad \frac{\partial^2 \phi_k}{\partial t\,\partial \xi_i}$$

throughout the domain

$$t_0 - e_0 \leqq t \leqq t_1 + e_1, \qquad |\tau - \tau_0| \leqq d, \qquad |\xi_k - \xi_k^0| \leqq d \quad (k = 1, 2, \cdots, n),$$

and that the Jacobian

$$\frac{\partial(\phi_1, \phi_2, \cdots, \phi_n)}{\partial(\xi_1, \xi_2, \cdots, \xi_n)}$$

is different from zero throughout the same domain.

§ 3. *The differential equations of the problem and their general integral.*

After these preliminaries we turn to the problem stated in the introduction.

We suppose the functions $f, f_1, \cdots, f_r$ to be of class C'' in a certain region **T** of the $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$-space. According to EULER-LAGRANGE's rule† we set

$$\Omega = f + \sum_{\rho=1}^{r} \lambda_\rho f_\rho,$$

where the λ's are unknown functions of x, and write for brevity

$$(15) \qquad \Omega_i = \frac{\partial \Omega}{\partial y_i}, \qquad \Omega_{n+i} = \frac{\partial \Omega}{\partial y_i'}, \qquad \Omega_{2n+\rho} = \frac{\partial \Omega}{\partial \lambda_\rho} \equiv f_\rho.$$

We start in our developments from the assumption that we have found a solution

$$(16) \quad y_i = y_i(x), \quad \lambda_\rho = \lambda_\rho(x), \quad x_0 \leqq x \leqq x_1 \qquad (i = 1, 2, \cdots, n;\ \rho = 1, 2, \cdots, r)$$

* Compare BLISS, loc. cit., pp. 61, 63, 67.

† Compare A. MAYER, Mathematische Annalen, vol. 26 (1886), p. 74, and Leipziger Berichte, 1895, p. 129; TURKSMA, Mathematische Annalen, vol. 47 (1896), p. 33; KNESER, *Lehrbuch der Variationsrechnung*, § 56–58; HAHN, Monatshefte für Mathematik und Physik, vol. 14 (1903), p. 326; HILBERT, Göttinger Nachrichten (1905), p. 1; compare also KNESER's article in the *Encyclopædie*, IIA, p. 579, 580.

of EULER-LAGRANGE's differential equations, i. e., of the system of $n + r$ differential equations

$$\text{(I)} \qquad \Omega_i - \frac{d}{dx}\Omega_{n+i} = 0, \qquad f_\rho = 0 \quad (i = 1, 2, \cdots, n\,;\ \rho = 1, 2, \cdots, r),$$

of the following properties:

A) the functions $y_i(x)$ are *of class* C'', the functions $\lambda_\rho(x)$ *of class* C' in the interval $(x_0 x_1)$.

B) the curve

$$y_i = y_i(x), \qquad y_i' = y_i'(x), \qquad x_0 \leqq x \leqq x_1 \quad (i = 1, 2, \cdots, n),$$

in the $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$-space lies *in the interior of the region* **T**.

C) the *Jacobian*

$$D(x, y_1, \cdots, y_n;\ y_1', \cdots, y_n';\ \lambda_1, \cdots, \lambda_r) = \frac{\partial(\Omega_{n+1}, \cdots, \Omega_{2n}, f_1, \cdots, f_r)}{\partial(y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_r)}$$

is *different from zero* along the solution (16), i. e.,

$$D[x, y_1(x), \cdots;\ y_1'(x), \cdots;\ \lambda_1(x), \cdots] \neq 0$$

throughout the interval $(x_0 x_1)$.

With the system (I) is associated a "canonical system"* of differential equations which is obtained as follows:

We apply lemma II to the problem of solving the $n + r$ equations†

$$\text{(17)} \quad \Omega_{n+i}(x, y_1, \cdots;\ y_1', \cdots;\ \lambda_1, \cdots) = v_i\,;\ f_\rho(x, y_1, \cdots;\ y_1', \cdots) = 0$$

with respect to

$$y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_r.$$

To the region **A** of the lemma corresponds in the present case the region

$$(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \text{ in } \mathbf{T};\ -\infty < \lambda_\rho < +\infty;\ -\infty < v_i < +\infty,$$

and to the point set **C** the curve

$$\text{(18)} \qquad y_i = y_i(x),\ y_i' = y_i'(x),\ \lambda_\rho \doteq \lambda_\rho(x),\ v_i = v_i(x),\ x_0 \leqq x \leqq x_1,$$

where $v_i(x)$ is defined by

$$\text{(19)} \qquad v_i(x) = \Omega_{n+i}[x, y_1(x), \cdots;\ y_1'(x), \cdots;\ \lambda_1(x), \cdots].$$

On account of our assumptions *A*), *B*), *C*) the curve (18) satisfies all the con-

*Compare KNESER's article in the *Encyclopædie*, IIA, pp. 584-586 and the references there given; also JORDAN, *Cours d'Analyse*, vol. 3, no. 375, and A. MAYER's papers referred to in the introduction and his paper in vol. 69 of the Journal für Mathematik, p. 240; we adopt, as far as possible, MAYER's notation.

† Here and in the sequel the index i (and likewise j, k) is always understood to run from 1 to n, the index ρ from 1 to r.

ditions required by the lemma. Hence there exist two positive quantities ϵ, δ, such that for every $x, y, \cdots, y_n, v_1, \cdots, v_n$ in the vicinity (δ) of the curve

$$y_i = y_i(x), \qquad v_i = v_i(x), \qquad x_0 \leqq x \leqq x_1, \tag{20}$$

the equations (17) admit one and but one solution $y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_n$ for which the point

$$(x, y_1, \cdots, y_n, y_1', \cdots, y_n', \lambda_1, \cdots, \lambda_r, v_1, \cdots, v_n)$$

lies in the vicinity (ϵ) of the curve (17). We denote this solution by

$$\begin{aligned} y_i' &= \Psi_i(x, y_1, \cdots, y_n;\ v_1, \cdots, v_n), \\ \lambda_\rho &= \Pi_\rho(x, y_1, \cdots, y_n;\ v_1, \cdots, v_n), \end{aligned} \tag{21}$$

so that, identically in (δ),

$$\begin{aligned} \Omega_{n+i}(x, y_1, \cdots;\ \Psi_1, \cdots;\ \Pi_1, \cdots) &\equiv v_i, \\ f_\rho(x, y_1, \cdots;\ \Psi_1, \cdots) &\equiv 0, \end{aligned} \tag{22}$$

and the system of values $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ thus obtained lies in the interior of the region $\mathbf{T}$. In particular

$$\begin{aligned} y_i'(x) &= \Psi_i[x, y_1(x), \cdots;\ v_1(x), \cdots], \\ \lambda_\rho(x) &= \Pi_\rho[x, y_1(x), \cdots;\ v_1(x), \cdots]. \end{aligned} \tag{23}$$

The first of these equations, combined with (19) and (I), shows that *the functions*

$$y_1(x), \cdots, y_n(x), v_1(x), \cdots, v_n(x)$$

satisfy the following system of $2n$ *differential equations:*

$$\begin{aligned} \frac{dy_i}{dx} &= \Psi_i(x, y_1, \cdots, y_n;\ v_1, \cdots, v_n), \\ \frac{dv_i}{dx} &= \Omega_i(x, y_1, \cdots, y_n;\ \Psi_1, \cdots, \Psi_n;\ \Pi_1, \cdots, \Pi_n). \end{aligned} \tag{II}$$

These differential equations form a so-called *canonical system.* For, if we denote by

$$H(x, y_1, \cdots, y_n, v_1, \cdots, v_n)$$

that function of $x, y_1, \cdots, y_n, v_1, \cdots, v_n$ into which the expression

$$\sum_i y_i' \Omega_{n+i}(x, y_1, \cdots;\ y_1', \cdots;\ \lambda_1, \cdots) - \Omega(x, y_1, \cdots;\ y_1', \cdots;\ \lambda_1, \cdots)$$

is transformed by the substitution (21), i. e.,

$$(24)\quad H(x, y_1, \cdots, v_1, \cdots) = \sum_i v_i \Psi_i - \Omega(x, y_1, \cdots;\ \Psi_1, \cdots;\ \Pi_1, \cdots),$$

and apply the identities (22), we easily obtain the result

$$\frac{\partial H}{\partial y_k} = -\Omega_k(x, y_1, \cdots;\ \Psi_1, \cdots;\ \Pi_1, \cdots),$$

$$\frac{\partial H}{\partial v_k} = \Psi_k(x, y_1, \cdots;\ v_1, \cdots).$$

Hence the system (II) may also be written

$$(\mathrm{II})\qquad \frac{dy_i}{dx} = \frac{\partial H}{\partial v_i}, \qquad \frac{dv_i}{dx} = -\frac{\partial H}{\partial y_i},$$

which is the characteristic form of a canonical system.

On account of (21_2), the expression for H may also be written

$$(24_a)\qquad H = \sum_i v_i \Psi_i - f(x, y_1, \cdots, \Psi_1, \cdots).$$

The functions on the right-hand side of (II) and the solution (20) of the canonical system (II) fulfil the conditions of lemma III. Hence we can take two quantities $X_0 < x_0$ and $X_1 > x_1$ so near to x_0 and x_1 respectively, and choose the positive quantity d so small that the following statements are true:

Let a^0 be an arbitrary value of x in the interval $(X_0 X_1)$ and write

$$y_i(a^0) = a_i^0, \qquad v_i(a^0) = b_i^0;$$

then if we choose

$$|a - a^0| \leqq d, \qquad |a_i - a_i^0| \leqq d, \qquad |b_i - b_i^0| \leqq d,$$

there exists a unique solution

$$(25)\qquad \begin{aligned} y_i &= \mathfrak{Y}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) \\ v_i &= \mathfrak{V}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) \end{aligned} \qquad (X_0 \leqq x \leqq X_1)$$

of the system (II) having the following properties:

a) the functions $\mathfrak{Y}_i$, $\mathfrak{V}_i$, considered as functions of their $2n + 2$ arguments, as well as their derivatives indicated by the operators

$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial a},\ \frac{\partial}{\partial a_k},\ \frac{\partial}{\partial b_k},\ \frac{\partial^2}{\partial x \partial a},\ \frac{\partial^2}{\partial x \partial a_k},\ \frac{\partial^2}{\partial x \partial b_k},\ \frac{\partial^2}{\partial x^2}\ ,$$

exist and are continuous throughout the domain

$$(26)\qquad X_0 \leqq x \leqq X_1,\ |a - a^0| \leqq d,\ |a_i - a_i^0| \leqq d,\ |b_i - b_i^0| \leqq d.$$

* The existence and continuity of $\partial^2 \mathfrak{Y}_i / \partial x^2$, $\partial^2 \mathfrak{Y} / \partial x^2$ follow from the fact that the functions on right-hand side of (II) are of class C' in (δ), which is slightly more than has been assumed in lemma III.

b) they satisfy the initial conditions

$$\begin{aligned} \mathfrak{Y}_i(a;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) &= a_i, \\ \mathfrak{V}_i(a:\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) &= b_i. \end{aligned} \tag{27}$$

c) the solution (25) lies in the vicinity (δ) of the curve (20).

d) for

$$a = a^0, \qquad a_i = a_i^0, \qquad b_i = b_i^0$$

the solution (25) coincides with the solution (20):

$$\begin{aligned} y_i(x) &\equiv \mathfrak{Y}_i(x;\ a^0, a_1^0, \cdots;\ b_1^0, \cdots), \\ v_i(x) &\equiv \mathfrak{V}_i(x;\ a^0, a_1^0, \cdots;\ b_1^0, \cdots). \end{aligned} \tag{28}$$

e) the Jacobian

$$\frac{\partial(\mathfrak{Y}_1, \cdots, \mathfrak{Y}_n, \mathfrak{V}_1, \cdots, \mathfrak{V}_n)}{\partial(a_1, \cdots, a_n, b_1, \cdots, b_n)}$$

is different from zero throughout the domain (26).

From the solution (25) of the system (II) a corresponding solution of the original system (I) is immediately obtained. For, from the definition of $\mathfrak{Y}_i$, $\mathfrak{V}_i$, it follows that *

$$\mathfrak{Y}_i' = \Psi_i(x, \mathfrak{Y}_1, \cdots;\ \mathfrak{V}_1, \cdots); \tag{29}$$

hence if we define

$$\mathfrak{L}_\rho(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) = \Pi_\rho(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_n;\ \mathfrak{V}_1, \cdots, \mathfrak{V}_n), \tag{30}$$

it follows from (II_2) that

$$\mathfrak{V}_i' = \Omega_i(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_1', \cdots, \mathfrak{L}_1, \cdots). \tag{31}$$

On the other hand, if in the identities (22) we replace y_i and v_i by $\mathfrak{Y}_i$ and $\mathfrak{V}_i$, respectively, and make use of (29) and (30) we obtain:

$$\Omega_{n+i}(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_1', \cdots, \mathfrak{L}_1, \cdots) \equiv \mathfrak{V}_i, \qquad f_\rho(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_1', \cdots) \equiv 0. \tag{32}$$

Combining these equations with (31), we obtain the result:

The functions

$$\begin{aligned} y_i &= \mathfrak{Y}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n), \\ \lambda_\rho &= \mathfrak{L}_\rho(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n), \end{aligned} \tag{33}$$

satisfy Euler-Lagrange's differential equations (*I*) *throughout the domain* (26).

* The accent always denotes differentiation with respect to x, even when other variables are present.

§ 4. *Hamilton's Principal Function.*

We consider now the integral

$$J = \int_{\mathfrak{C}} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx$$

taken along the "extremal" *

$$\mathfrak{E}: \qquad y_i = \mathfrak{Y}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n)$$

from the point whose abscissa is a to the point whose abscissa is x.

We denote the value of this integral considered as a function of $x, a, a_1, \cdots, a$; $b_1, \cdots, b_n$ by

$$\mathfrak{U}(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n)$$

or simply $\mathfrak{U}$, so that

$$\mathfrak{U} = \int_a^x f(x, \mathfrak{Y}_1, \cdots;\ \mathfrak{Y}_1', \cdots)\, dx, \tag{34}$$

or as we may also write on account of (32_2), (24_a) and (29):

$$\mathfrak{U} = \int_a^x \Omega(x, \mathfrak{Y}_1, \cdots, \mathfrak{Y}_1', \cdots, \mathfrak{Y}_1, \cdots)\, dx, \tag{34_a}$$

or

$$\mathfrak{U} = \int_a^x [\sum_i \mathfrak{V}_i \mathfrak{Y}_i' - H(x, \mathfrak{Y}_1, \cdots;\ \mathfrak{V}_1, \cdots)]\, dx. \tag{34_b}$$

We propose to compute the partial derivatives of $\mathfrak{U}$; for this purpose the expression (34_b) is most convenient.† We obtain immediately

$$\frac{\partial \mathfrak{U}}{\partial x} = \sum_i \mathfrak{V}_i \mathfrak{Y}_i' - H(x, \mathfrak{Y}_1, \cdots;\ \mathfrak{V}_1, \cdots). \tag{35}$$

Further, if we denote by α any one of the quantities a_k, b_k:

$$\frac{\partial \mathfrak{U}}{\partial \alpha} = \int_a^x \sum_i \left(\frac{\partial \mathfrak{V}_i}{\partial \alpha} \mathfrak{Y}_i' + \mathfrak{V}_i \frac{\partial \mathfrak{Y}_i'}{\partial \alpha} - H_i \frac{\partial \mathfrak{Y}_i}{\partial \alpha} - H_{n+i} \frac{\partial \mathfrak{V}_i}{\partial \alpha} \right) dx,$$

where

$$H_i = \frac{\partial H}{\partial y_i}, \qquad H_{n+i} = \frac{\partial H}{\partial v_i}, \tag{36}$$

and the arguments of H, H_i, H_{n+i} are x, $\mathfrak{Y}_1, \cdots, \mathfrak{Y}_n$, $\mathfrak{V}_1, \cdots, \mathfrak{V}_n$.

* We call *extremal* every set of functions $y_1, \cdots, y_n$ of x with which can be associated a system of functions $\lambda_1, \cdots, \lambda_r$ such that the functions y_i, λ_ρ constitute a solution of (I).

† With (34_a) the computation is slightly more complicated.

Applying to the second term under the integral sign integration by parts, we obtain

$$\frac{\partial \mathfrak{U}}{\partial \alpha} = \left[\sum_i \mathfrak{B}_i \frac{\partial \mathfrak{Y}_i}{\partial \alpha} \right]_a^x + \int_a^x \sum_i \left(\frac{\partial \mathfrak{B}_i}{\partial \alpha} (\mathfrak{Y}_i)' - H_{n+i} \right) - \frac{\partial \mathfrak{Y}_i}{\partial \alpha} (\mathfrak{B}_i' + H_i) \Big) dx .$$

But the functions $\mathfrak{Y}_i$, $\mathfrak{B}_i$ satisfy the system (II); hence the integral is zero and we obtain on account of (27):

$$\frac{\partial \mathfrak{U}}{\partial \alpha} = \sum_i \mathfrak{B}_i \frac{\partial \mathfrak{Y}_i}{\partial \alpha} - \sum_i b_i \left(\frac{\partial \mathfrak{Y}_i}{\partial \alpha} \right)_{x=a} .$$

Taking successively $\alpha = a_k$, b_k and making use of the relations

$$\left(\frac{\partial \mathfrak{Y}_i}{\partial a_k} \right)_{x=a} = \delta_{ik}, * \qquad \left(\frac{\partial \mathfrak{Y}_i}{\partial b_k} \right)_{x=a} = 0 , \tag{37}$$

derived from (27) by differentiation with respect to a_k, b_k, we obtain the result

$$\frac{\partial \mathfrak{U}}{\partial a_k} = \sum_i \mathfrak{B}_i \frac{\partial \mathfrak{Y}_i}{\partial a_k} - b_k , \tag{38}$$

$$\frac{\partial \mathfrak{U}}{\partial b_k} = \sum_i \mathfrak{B}_i \frac{\partial \mathfrak{Y}_i}{\partial b_k} . \tag{39}$$

We now introduce, in addition to the assumptions A), B), C) of § 3, the new assumption concerning the solution (16) of the original system (I) that the Jacobian †

* δ_{ik} is "KRONECKER's symbol," viz. $\delta_{ik} = 0$ if $i \neq k$, $\delta_{ii} = 1$.

† We notice incidentally that on account of (37_2) the Jacobian Δ is (apart from the sign) identical with MAYER's determinant

$$\Delta(x, a) = \begin{vmatrix} \frac{\partial \mathfrak{Y}_i}{\partial a_1}, & \cdots, & \frac{\partial \mathfrak{Y}_i}{\partial a_n}, & \frac{\partial \mathfrak{Y}_i}{\partial b_1}, & \cdots, & \frac{\partial \mathfrak{Y}_i}{\partial b_n} \\ \left(\frac{\partial \mathfrak{Y}_i}{\partial a_1} \right)_{x=a}, \cdots, \left(\frac{\partial \mathfrak{Y}_i}{\partial a_n} \right)_{x=a}, & \left(\frac{\partial \mathfrak{Y}_i}{\partial b_1} \right)_{x=a}, \cdots, \left(\frac{\partial \mathfrak{Y}_i}{\partial b_n} \right)_{x=a} \end{vmatrix} \quad (i = 1, 2, \ldots, n),$$

which furnishes "JACOBI's criterion" for the problem under consideration [see A. MAYER, Journal für Mathematik, vol. 69 (1868), p. 250]. Hence it follows that it is possible to choose a^0 so that D) is satisfied, whenever "JACOBI's condition" is fulfilled (including the end-point x_1); see JORDAN, *Cours d'Analyse*, vol. 3, no. 393.

As to the possibility of an identical vanishing of the determinant $\Delta(x, a)$, compare MAYER, Journal für Mathematik, vol. 69, p. 251; *Encyclopædie*, IIA, p. 599 (KNESER), p. 634 (ZERMELO and HAHN) and the references to VON ESCHERICH given there.

We also notice that the assumption D) implies that a^0 lies outside of the interval (x_0, x_1), since on account of (37_3)

$$\Delta(a^0; a^0, a_1^0, \cdots; b_1^0) = 0 .$$

According to its definition, a^0 must therefore be taken in one of the two intervals

$$X_0 \leqq a^0 < x_0 , \qquad \text{or} \qquad x_1 < a^0 \leqq X_1 .$$

$$\Delta(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n) = \frac{\partial(\mathfrak{Y}_1, \cdots, \mathfrak{Y}_n)}{\partial(b_1, \cdots, b_n)}$$

is different from zero along the particular extremal

$$\mathfrak{E}_0: \qquad y_i = y_i(x) \equiv \mathfrak{Y}_i(x;\ a^0, a_1^0, \cdots;\ b_1^0, \cdots) \qquad (x_0 \leqq x \leqq x_1),$$

i. e., that

D) $\qquad \Delta(x;\ a^0, a_1^0, \cdots;\ b_1^0, \cdots) \neq 0 \qquad$ for $\qquad x_0 \leqq x \leqq x_1.$

It follows then from our lemma III* that two positive quantities k and l $(l \leqq k)$ can be assigned such that for every point $x, y_1, \cdots, y_n$ in the vicinity (l) of the extremal

$$\mathfrak{E}_0: \qquad y_i = y_i(x), \qquad x_0 \leqq x \leqq x_1,$$

and for every a, a_i such that

$$|a - a^0| < l, \qquad |a_i - a_i^0| < l,$$

the n equations

(40) $\qquad y_i = \mathfrak{Y}_i(x;\ a, a_1, \cdots, a_n;\ b_1, \cdots, b_n)$

admit one and but one solution $b_1, \cdots, b_n$ for which

$$|b_i - b_i^0| < k.$$

The corresponding inverse functions

(41) $\qquad b_i = \mathfrak{B}_i(x, y_1, \cdots, y_n;\ a, a_1, \cdots, a_n)$

are of class C' in the domain

(42) $\qquad (x, y_1, \cdots, y_n)$ in $(l)_{\mathfrak{E}_0}; \qquad |a - a^0| < l, \qquad |a_i - a_i^0| < l;$

they satisfy in this domain the identities

(43) $\qquad \mathfrak{Y}_i(x;\ a, a_1, \cdots;\ \mathfrak{B}_1, \cdots) \equiv y_i.$

If we replace in $\mathfrak{U}(x;\ a, a_1, \cdots;\ b_1, \cdots)$ the b_i's by the function $\mathfrak{B}_i$, the function $\mathfrak{U}$ is transformed into a function of $x, y_1, \cdots, y_n;\ a, a_1, \cdots, a_n$; we denote it by

(44) $\mathfrak{W}(x, y_1, \cdots, y_n;\ a, a_1, \cdots, a_n) \equiv \mathfrak{U}(x;\ a, a_1, \cdots, a_n;\ \mathfrak{B}_1, \cdots, \mathfrak{B}_n).$

It represents *the value of the integral J taken along the uniquely defined extremal joining the two points* $(a, a_1, \cdots, a_n)$ *and* $(x, y_1, \cdots, y_n)$, *considered as a function of the coördinates of these two points.* It is identical with

* To the point set C of the lemma corresponds here the set

$$x_0 \leqq x \leqq x_1,\ y_i = y_i(x),\ a = a^0,\ a_i = a_i^0,\ b_i = b_i^0.$$

HAMILTON'S *Principal Function** and a generalization of DARBOUX'S *Geodesic Distance.*†

From the identities (43) we obtain by differentiation the relations

$$\frac{\partial \overline{\mathfrak{Y}}_i}{\partial a_k} + \sum_j \frac{\partial \overline{\mathfrak{Y}}_i}{\partial b_j}\frac{\partial \mathfrak{B}_j}{\partial a_k} = 0,$$

$$\frac{\partial \overline{\mathfrak{Y}}_i}{\partial x} + \sum_j \frac{\partial \overline{\mathfrak{Y}}_i}{\partial b_j}\frac{\partial \mathfrak{B}_j}{\partial x} = 0, \tag{45}$$

$$\sum_j \frac{\partial \overline{\mathfrak{Y}}_i}{\partial b_j}\frac{\partial \mathfrak{B}_j}{\partial y_k} = \delta_{ik},$$

where we have indicated by a stroke the substitution of $\mathfrak{B}_i$ for b_i.

If we combine the relations (45) with the previous results (35), (38), (39), we obtain the following theorems due to HAMILTON:*

The partial derivatives of Hamilton's "Principal Function," defined by the equations (34) *and* (44), *have the following values*

$$\frac{\partial \mathfrak{W}}{\partial x} = -H(x, y_1, \cdots, y_n;\ \bar{\mathfrak{B}}_1, \cdots, \bar{\mathfrak{B}}_n),$$

$$\frac{\partial \mathfrak{W}}{\partial y_k} = \bar{\mathfrak{B}}_k, \qquad \frac{\partial \mathfrak{W}}{\partial a_k} = -\mathfrak{B}_k, \tag{46}$$

where

$$\bar{\mathfrak{B}}_k = \mathfrak{B}_k(x;\ a, a_1, \cdots, a_n;\ \mathfrak{B}_1, \cdots, \mathfrak{B}_n).$$

An immediate consequence (which we shall however not need in the sequel) is the well-known theorem that $\mathfrak{W}$ satisfies "HAMILTON'S partial differential equation":

$$\frac{\partial \mathfrak{W}}{\partial x} + H\left(x, y_1, \cdots, y_n, \frac{\partial \mathfrak{W}}{\partial y_1}, \cdots, \frac{\partial \mathfrak{W}}{\partial y_n}\right) = 0.‡$$

§ 5. *Weierstrass' theorem for a set of extremals through a fixed point.*

We now give the quantities a, a_i the special values

$$a = a^0,\ a_i = a_i^0,$$

* Philosophical Transactions, 1835, part I, p. 99.

† *Theorie des Surfaces*, vol. 2, no. 536.

‡ All these results might have been taken directly from HAMILTON'S theory, for instance in E. VON WEBER'S presentation in his *Vorlesungen über das Pfaff'sche Problem*, art. 386. But on the one hand, I did not wish to presuppose HAMILTON'S theory, on the other hand the question of the inverse functions $\mathfrak{B}_i$ requires a slightly different treatment from the Calculus of Variations standpoint.

and write for brevity

$$\mathfrak{Y}_i(x;\ a^0, a_1^0, \cdots, a_n^0, b_1, \cdots, b_n) = \mathfrak{y}_i(x, b_1, \cdots, b_n),$$
$$\mathfrak{V}_i(x;\ a^0, a_1^0, \cdots, a_n^0, b_1, \cdots, b_n) = \mathfrak{v}_i(x, b_1, \cdots, b_n),$$
$$\mathfrak{L}_\rho(x;\ a^0, a_1^0, \cdots, a_n^0, b_1, \cdots, b_n) = \mathfrak{l}_\rho(x, b_1, \cdots, b_n).$$

Then the equations

$$y_i = \mathfrak{y}_i(x;\ b_1, \cdots, b_n), \tag{47}$$

in which $b_1, \cdots, b_n$ are considered as variable parameters, represent the set of extremals through the point

$$x = a^0, y_1 = a_1^0, \cdots, y_n = a_n^0,$$

and from what has been said in § 4 about the solution of the equations (40), it follows a fortiori, that for every $x, y_1, \cdots, y_n$ in the vicinity $(l)_{\mathfrak{E}_0}$ the equations (47) admit one and but one solution $b_1, \cdots, b_n$ for which $|b_i - b_i^0| < k$. The vicinity $(l)_{\mathfrak{E}_0}$ constitutes therefore a *field of extremals about the extremal* $\mathfrak{E}_0$.

The corresponding inverse functions are

$$b_i = \mathfrak{B}_i(x, y_1, \cdots, y_n, a^0, a_1^0, \cdots, a_n^0) \equiv \mathfrak{b}_i(x, y_1, \cdots, y_n).$$

Further we define

$$\mathfrak{w}(x, y_1, \cdots, y_n) = \mathfrak{W}(x, y_1, \cdots, y_n;\ a^0, a_1^0, \cdots, a_n^0). \tag{48}$$

The function $\mathfrak{w}(x, y_1, \cdots, y_n)$ is identical with the value of the integral J taken from the fixed point $(a^0, a_1^0, \cdots, a_n^0)$ to the point $(x, y_1, \cdots, y_n)$ of the field along the unique extremal of the field passing through the point $(x, y_1, \cdots, y_n)$; we shall call it the *field integral*. From (46) it follows that its partial derivatives have the values

$$\frac{\partial \mathfrak{w}}{\partial x} = -H(x, y_1, \cdots, y_n;\ \bar{\mathfrak{v}}_1, \cdots, \bar{\mathfrak{v}}_n), \qquad \frac{\partial \mathfrak{w}}{\partial y_k} = \bar{\mathfrak{v}}_k, \tag{49}$$

where the stroke indicates the substitution of $\mathfrak{b}_i$ for b_i. These expressions may be thrown into a different form by introducing the functions

$$p_i(x, y_1, \cdots, y_n) = \mathfrak{y}_i'(x;\ \mathfrak{b}_1, \cdots, \mathfrak{b}_n),$$
$$\mu_\rho(x, y_1, \cdots, y_n) = \mathfrak{l}_\rho(x;\ \mathfrak{b}_1, \cdots, \mathfrak{b}_n). \tag{50}$$

For, according to (29), (30) and (32),

$$\mathfrak{v}_i = \Omega_{n+i}(x, \mathfrak{y}_1, \cdots;\ \mathfrak{y}_i', \cdots;\ \mathfrak{l}_1, \cdots),$$
$$\mathfrak{y}_i' = \Psi_i(x, \mathfrak{y}_1, \cdots;\ \mathfrak{v}_1, \cdots),$$
$$\mathfrak{l}_\rho = \Pi_\rho(x, \mathfrak{y}_1, \cdots;\ \mathfrak{v}_1, \cdots)$$

and therefore on replacing b_i by $\mathfrak{b}_i$, and applying (50), we get

$$\bar{v}_i = \Omega_{n+i}(x, y_i, \cdots; p_1, \cdots; \mu_1, \cdots),$$
$$p_i = \Psi_i(x, y_1, \cdots; \bar{v}_1, \cdots),$$
$$\mu_\rho = \Pi_\rho(x, y_1, \cdots; \bar{v}_1, \cdots).$$

Substituting finally for H its value from (24_a), we obtain the result:

The partial derivatives of the field-integral $\mathfrak{w}$ *have the following values:*

$$\frac{\partial \mathfrak{w}}{\partial x} = f(x, y_1, \cdots; p_1, \cdots) - \sum_i p_i \Omega_{n+i}(x, y_1, \cdots; p_1, \cdots; \mu_1, \cdots),$$
$$\frac{\partial \mathfrak{w}}{\partial y_k} = \Omega_{n+k}(x, y_1, \cdots; p_1, \cdots; \mu_1, \cdots). \tag{51}$$

Let now

$$\tilde{\mathfrak{E}}: \qquad y_i = \tilde{y}_i(x), \qquad x_0 \leqq x \leqq x_1$$

be any curve of class C' which satisfies the differential equations (5), lies in the field $(l)_{\mathfrak{E}_0}$, and passes through the two end points of $\mathfrak{E}_0$, viz.,

$$P_0(x_0, y_{10}, \cdots, y_{n0}) \qquad \text{and} \qquad P_1(x_1, y_{11}, \cdots, y_{n1}),$$

where

$$y_{i0} = y_i(x_0), \qquad y_{i1} = y_i(x_1).$$

To this curve we can apply WEIERSTRASS' *construction.* In order to fix the ideas we choose* $a^0 < x_0$. Then, if $P(x, y_1, \cdots, y_n)$ is any point of $\tilde{\mathfrak{E}}$, we draw the unique extremal $\mathfrak{E}$ of the field from P^0 to P and consider the function

$$S(x) = J_{\mathfrak{E}}(P^0P) + J_{\tilde{\mathfrak{E}}}(PP_1).$$

We have then, in the usual manner,

$$\Delta J = J_{\tilde{\mathfrak{E}}} - J_{\mathfrak{E}_0} = -[S(x_1) - S(x_0)].$$

But

$$J_{\mathfrak{E}}(P^0P) = \mathfrak{w}[x, \tilde{y}(x), \cdots, \tilde{y}_n(x)],$$
$$J_{\tilde{\mathfrak{E}}}(PP_1) = \int_x^{x_1} f[x, \tilde{y}_1(x), \cdots, \tilde{y}_n(x), \tilde{y}_1'(x), \cdots, \tilde{y}_n'(x)]\, dx.$$

Hence we obtain for the derivative $S'(x)$:

$$S'(x) = \frac{\partial \mathfrak{w}}{\partial x} + \sum_i \frac{\partial \mathfrak{w}}{\partial y_i} \tilde{y}_i' - f(x, y_1, \cdots, y_n, \tilde{y}_1', \cdots, \tilde{y}_n'),$$

that is, according to (51),

$$S'(x) = -\sum_i (p_i - \tilde{y}_i')\Omega_{n+i}(x, y_1, \cdots, p_1, \cdots, \mu_1, \cdots) + f(x, y_1, \cdots, p_1, \cdots) - f(x, y_1, \cdots, \tilde{y}_1', \cdots).$$

* Compare the footnote on p. 473.

Hence if we introduce the E-function by the definition

$$E(x, y_1, \cdots, y_n;\ p_1, \cdots, p_n;\ \tilde{y}'_1, \cdots, \tilde{y}'_n;\ \mu_1, \cdots, \mu_r) = f(x, y_1, \cdots, \tilde{y}'_1, \cdots) - f(x, y_1, \cdots, p_1, \cdots) - \sum_i (\tilde{y}'_i - p_i)\Omega_{n+i}(x, y_1, \cdots, p_1, \cdots, \mu_1, \cdots), \tag{52}$$

we obtain *Weierstrass' Theorem*:

$$\Delta J = \int_{x_0}^{x_1} E\{x, \tilde{y}_1(x), \cdots;\ p_1[x, y_1(x), \cdots], \cdots;\ \tilde{y}'_1(x) \cdots;\ \mu_1[x, \tilde{y}_1(x), \cdots], \cdots\} dx, \tag{53}$$

in which the functions p_i, μ_ρ are defined by (50).

§ 6. *Kneser's Problem generalized.*

In the preceding section WEIERSTRASS' theorem has been proved for the special case of an n-parametric set of extremals passing through a fixed point; in this and the following sections, we propose to extend the theorem to a more general class of n-parametric sets. The method which we shall use will be entirely analogous to the method by which KNESER* has accomplished the same result for the simplest case $n = 1$, $r = 0$.

Accordingly, let

$$y_i = Y_i(x, a_1, \cdots, a_n), \qquad v_i = V_i(x, a_1, \cdots, a_n), \tag{54}$$

be any n-parametric set of solutions of the differential equations (II), containing the special solution

$$y_i = y_i(x), \qquad v_i = v_i(x), \tag{20}$$

say for $a_i = a_i^0$, and furnishing a field † $\mathbf{S}_k$ of extremals about the particular extremal $\mathfrak{E}_0$. The inverse functions of the field will be denoted by

$$a_i = \mathfrak{a}_i(x, y_1, \cdots, y_n)$$

so that identically

$$Y_i(x;\ \mathfrak{a}_1, \cdots, \mathfrak{a}_n) \equiv y_i. \tag{56}$$

Across the field $\mathbf{S}_k$ we draw an n-dimensional surface which meets every extre-

* Compare the Introduction.

† We include in this assumption the condition that the functions Y_i, $\partial Y_i/\partial x$ shall be of class C' in a certain domain

$$x_0 - \delta < x < x_1 + \delta, \qquad |a_i - a_i^0| < \delta.$$

The field $\mathbf{S}_k$ consists of all points $x, y_1, \cdots, y_n$ furnished by the equations

$$y_i = Y_i(x;\ a_1, \cdots, a_n), \tag{55}$$

when $x, a_1, \cdots, a_n$ are restricted to the domain

$$x_0 - k < x < x_1 + k, \qquad |a_i - a_i^0| < k.$$

mal of the field in one and but one point; such a surface may be represented, in parameter-representation, in the form

$$x = \xi(a_1, \cdots, a_n), \qquad y_i = Y_i(\xi;\ a_1, \cdots, a_n). \tag{57}$$

Then we consider our integral J taken along the extremal $\mathfrak{E}: a_1, \cdots, a_n$ of the set (55) from its point of intersection with the surface (57), i. e., from the point P^0 whose abscissa is $\xi(a_1, \cdots, a_n)$, to the point P whose abscissa is x, i. e., the integral

$$U(x;\ a_1, \cdots, a_n) = \int_{\xi(a_1, \cdots, a_n)}^{x} f(x,\ Y_1, \cdots,\ Y_1', \cdots)dx. \tag{58}$$

The same integral, considered as a function of the coördinates $x, y_1, \cdots, y_n$ of the end point P will be denoted by $W(x, y_1, \cdots, y_n)$, so that

$$W(x, y_1, \cdots, y_n) = U(x;\ a_1, \cdots, a_n). \tag{59}$$

We call the surface (57) the "surface of reference" for the definition of the function W.

And now *we propose so to choose the surface* (57) *i. e., the function* $\xi(a_1, \cdots, a_n)$, *that the partial derivatives of the function* W *shall have the same simple form as in the previous case of a set of extremals through a fixed point, viz.,*

$$\begin{aligned} \frac{\partial W(x, y_1, \cdots, y_n)}{\partial x} &= -H(x, y_1, \cdots, y_n, \bar{Y}_1, \cdots,\ \bar{Y}_n), \\ \frac{\partial W(x, y_1, \cdots, y_n)}{\partial y_k} &= \bar{Y}_k, \end{aligned} \tag{60}$$

where

$$\bar{Y}_k = Y_k(x;\ a_1, \cdots, a_n).$$

As soon as this is accomplished WEIERSTRASS' theorem for the set (55) will follow immediately.

From the connection between the functions $W(x,\ y_1,\ \cdots,\ y_n)$ and $U(x;\ a_1,\ \cdots, a_n)$ it follows that the partial derivatives of W will take the above form if and only if the partial derivatives of U have the values

$$\frac{\partial U}{\partial x} = f(x, Y_1, \cdots, Y_1', \cdots), \qquad \frac{\partial U}{\partial a_k} = \sum_i Y_i \frac{\partial Y}{\partial a_k}, \tag{61}$$

corresponding to (35) and (39).

The expression for $\partial U/\partial x$ has always the desired form, however we may choose ξ; it remains, therefore, so to determine the function $\xi(a_1, \cdots, a_n)$ that (61_2) shall hold.

We start from the remark, which is easily verified, that every n-parametric set of solutions of the system (II), which furnishes a field of extremals about our

extremal $\mathfrak{E}_0$, can be thrown by a parameter-transformation, into the canonical form

$$(62)\qquad \begin{aligned} y_i &= \mathfrak{Y}_i(x;\ a^0, a_1, \cdots, a_n;\ B_1, \cdots, B_n),\\ v_i &= \mathfrak{V}_i(x;\ a^0, a_1, \cdots, a_n;\ B_1, \cdots, B_n), \end{aligned}$$

where $B_1, \cdots, B_n$ are functions of $a_1, \cdots, a_n$ of class C' in a certain vicinity of $a_1 = a_1^0, \cdots, a_n = a_n^0$, satisfying the initial conditions

$$(63)\qquad B_i(a_1^0, \cdots, a_n^0) = b_i^0;$$

the quantities $a^0, a_1^0, \cdots, a_n^0$ have the same signification as in § 3, and $a_1, \cdots, a_n$ are considered as the parameters of the set.

We may, without loss of generality, suppose that the parameters $a_1, \cdots, a_n$ of the set (54) are these canonical parameters, so that

$$(64)\qquad \begin{aligned} Y_i(x;\ a_1, \cdots, a_n) &\equiv \mathfrak{Y}_i(x;\ a^0, a_1, \cdots, a_n;\ B_1, \cdots, B_n),\\ V_i(x;\ a_1, \cdots, a_n) &\equiv \mathfrak{V}_i(a;\ a^0, a_1, \cdots, a_n;\ B_1, \cdots, B_n), \end{aligned}$$

whence it follows that

$$Y_i(a^0;\ a_1, \cdots, a_n) = a_i, \qquad V_i(a^0;\ a_1, \cdots, a_n) = B_i.$$

The integral $U(x;\ a_1, \cdots, a_n)$ may therefore be expressed in terms of the function $\mathfrak{U}$ defined by (34), viz:

$$U(x;\ a_1, \cdots, a_n) = \mathfrak{U}(x;\ a^0, a_1, \cdots:\ B_1, \cdots) - \mathfrak{U}(\xi;\ a^0, a_1, \cdots;\ B_1, \cdots).$$

Hence we obtain

$$\frac{\partial U}{\partial a_k} = \left(\frac{\partial \mathfrak{U}(x)}{\partial a_k}\right) + \sum_j \left(\frac{\partial \mathfrak{U}(x)}{\partial b_j}\right)\frac{\partial B_j}{\partial a_k} - \frac{\partial \mathfrak{U}(\xi)}{\partial a_k},$$

where the bracket ( ) indicates the substitution of B_i for b_i and of a^0 for a. Substituting the values of $\partial \mathfrak{U}/\partial a_k$, $\partial \mathfrak{U}/\partial b_k$ from (38) and (39), and noticing that according to (64)

$$\frac{\partial Y_i}{\partial a_k} = \left(\frac{\partial \mathfrak{Y}_i}{\partial a_k}\right) + \sum_j \left(\frac{\partial \mathfrak{Y}_i}{\partial b_j}\right)\frac{\partial B_j}{\partial a_k},$$

we obtain

$$(65)\qquad \frac{\partial U}{\partial a_k} = \sum_i V_i \frac{\partial Y_i}{\partial a_k} - B_k - \frac{\partial \mathfrak{U}(\xi)}{\partial a_k}.$$

In order that $\partial U/\partial a_k$ may have the desired form (61_2), it is therefore necessary and sufficient that

$$(66)\qquad B_k = -\frac{\partial \mathfrak{U}(\xi)}{\partial a_k}.$$

Hence we infer in the first place: *The desired determination of the function $\xi(a_1, \cdots, a_n)$ is, for $n > 1$, not possible for all n-parametric sets of ex-*

tremals, but only for those special sets for which in the normal form (62) *the functions* $B_1, \cdots, B_n$ *are the partial derivatives with respect to* $a_1, \cdots, a_n$ *respectively of one and the same function of* $a_1, \cdots, a_n$, *for which therefore the functions* B_i *satisfy the integrability condition*

$$\frac{\partial B_h}{\partial a_k} = \frac{\partial B_k}{\partial a_h}. \tag{67}$$

We shall call these special n-parametric sets of extremals *Mayerian sets of extremals*, because they are identical with the sets discovered by MAYER,* which have the peculiarity of furnishing solutions of HILBERT's Problem.

We suppose in the sequel that for our set (54) the condition (67) is fulfilled, so that

$$B_k = \frac{\partial A(a_1, \cdots, a_n)}{\partial a_k}, \tag{68}$$

where $A(a_1, \cdots, a_n)$ is any function of $a_1, \cdots, a_n$, of class C'' in the vicinity of $a_1^0, \cdots, a_n^0$, which satisfies according to (63) the initial condition

$$\frac{\partial A(a_1, \cdots, a_n)}{\partial a_k}\Big|_{a_i = a_i^0} = b_k^0. \tag{69}$$

Our condition (66) takes then the form

$$\mathfrak{U}\left(\xi;\ a^0, a_1, \cdots, a_n;\ \frac{\partial A}{\partial a_1}, \cdots, \frac{\partial A}{\partial a_n}\right) = -A(a_1, \cdots, a_n) + c, \tag{70}$$

c being a numerical constant independent of $a_1, \cdots, a_n$.

For the discussion of this equation we write for brevity

$$\mathfrak{U}\left(x;\ a^0, a_1, \cdots, a_n;\ \frac{\partial A}{\partial a_1}, \cdots, \frac{\partial A}{\partial a_n}\right) + A(a_1, \cdots, a_n) \equiv G(x;\ a_1, \cdots, a_n)$$

and add to our previous assumptions † A) to D) the further assumption

$$E) \qquad f[x, y_1(x), \cdots;\ y_1'(x), \cdots] \neq 0 \text{ in } (x_0 x_1).$$

Under this assumption we consider the problem of solving the equation

$$G(x;\ a_1, \cdots, a_n) = u \tag{71}$$

with respect to x.

* See MAYER II, §§ 2, 3; compare also below, § 8, end. From the fact that for the special sets of extremals through a fixed point the partial derivatives of the field integral have the simple form (49), it follows that these special sets, when properly normalized, satisfy the condition (67). Compare MAYER's remarks on this point, MAYER, II, p. 63.

† See pp. 468 and 474.

Since according to (69), (64), and (28)

$$Y_i(x;\ a_1^0, \cdots, a_n^0) \equiv y_i(x),$$

it follows from E) that

$$\frac{\partial G(x;\ a_1, \cdots, a_n)}{\partial x} \neq 0$$

for

$$x_0 \leqq x \leqq x_1, \qquad a_i = a_i^0,$$

and therefore

$$G(x';\ a_1^0, \cdots, a_n^0) \neq G(x'';\ a_1^0, \cdots, a_n^0),$$

if x', x'' are any two distinct values of x in the interval $(x_0 x_1)$. Hence on applying * lemma I of § 1 to the system of equations

$$u = G(x;\ a_1, \cdots, a_n), \qquad u_i = a_i,$$

we obtain the result that *the equation* (71) *admits a unique solution*

$$x = \xi(a_1, \cdots, a_n;\ u) \tag{72}$$

in the following sense: The positive quantity k can be taken so small that if x', a_1', $\cdots$, a_n' is any point in the domain

$$x_0 - k < x < x_1 + k, \qquad |a_i - a_i^0| < k,$$

and $G(x';\ a_1', \cdots, a_n') = u'$, then the equation

$$G(x;\ a_1', \cdots, a_n') = u'$$

has no other solution than $x = x' = \xi(a_1', \cdots, a_n';\ u')$, satisfying the condition

$$x_0 - k < x < x_1 + k.$$

Further, if G_0 and G_1 denote the minimum and maximum of the function $G(x;\ a_1^0, \cdots, a_n^0)$ in the interval $(x_0 x_1)$, it follows from the corollary to lemma I that a second positive quantity $l \leqq k$ can be determined so that for every $a_1, \cdots, a_n;\ u$ in the domain

$$G_0 - l < u < G_1 + l, \qquad |a_i - a_i^0| < l, \tag{73}$$

equation (71) has one and but one solution

$$x = \xi(a_1, \cdots, a_n;\ u)$$

for which $x_0 - k < x < x_1 + k$. Moreover the equation

$$G[\xi(a_1, \cdots, a_n;\ u), a_1, \cdots, a_n] = u \tag{74}$$

* To the point set C of the lemma corresponds here the set

$$x_0 \leqq x \leqq x_1, \quad a_i = a_i^0.$$

holds identically in the domain (73), and the function $\xi(a_1, \cdots, a_n; u)$ is of class C' in the same domain.

Hence follows the result: *If the functions B_i satisfy the condition* (67), *our problem has an infinitude of solutions, viz.,*

$$\xi = \xi(a_1, \cdots, a_n; c),$$

differing from each other by the value of the constant c. The latter may take any value between $G_0 - l$ and $G_1 + l$. If we take $\xi(a_1, \cdots, a_n; c)$ as lower limit in the definition of the function $U(x; a_1, \cdots, a_n)$, the partial derivatives of U have the desired values (61). The functions U corresponding to two different values c' and c'' of the constant c can therefore differ only by an additive constant independent of $x, a_1, \cdots, a_n$, a result which can immediately be verified directly, since, according to (74),

$$\begin{aligned}(75)\qquad &\int_{\xi(a_1, \cdots, a_n; c')}^{\xi(a_1, \cdots, a_n; c'')} f(x, Y_1, \cdots; Y_1', \cdots)dx \\ &\quad = G[\xi(a_1, \cdots, a_n; c''); a_1, \cdots, a_n] \\ &\qquad - G[\xi(a_1, \cdots, a_n; c'); a_1, \cdots, a_n] = c'' - c'.\end{aligned}$$

§ 7 *Kneser's theorem on transversals generalized.*

Passing now from the consideration of the function $U(x, a_1, \cdots, a_n)$ to the function $W(x, y_1, \cdots, y_n)$, we write for brevity

$$(76)\qquad Y_i[\xi(a_1, \cdots, a_n; u); a_1, \cdots, a_n] \equiv \eta_i(a_1, \cdots, a_n; u).$$

The equations

$$(77)\qquad x = \xi(a_1, \cdots, a_n; c), \qquad y_i = \eta_i(a_1, \cdots, a_n; c)$$

represent then, in parameter-representation, an n-dimensional surface $\mathfrak{T}_c$ in the $(n+1)$-dimensional space of the variables $x, y_1, \cdots, y_n$. It lies entirely in the field $\mathbf{S}_k$ if $a_1, \cdots, a_n$ are restricted to the domain: $|a_i - a_i^0| < l$; giving c different constant values we obtain a one-parameter set of surfaces, and from what has been said about the solution of the equation (71), it follows that through every point of the field $\mathbf{S}_k$ passes one and but one surface of the set.

We shall call these surfaces the "*transversal surfaces*" of the field $\mathbf{S}_k$ formed by the set of extremals

$$(55)\qquad y_i = Y_i(x; a_1, \cdots, a_n),$$

because they are the generalization of KNESER's "transversals." If we take the integral $W(x, y_1, \cdots, y_n)$ from the point of intersection of the extremal passing through the point $(x, y_1, \cdots, y_n)$ with a fixed transversal surface of reference,

the partial derivatives of W have the desired form (60). If we change the transversal surface of reference, the function $W(x, y_1, \cdots, y_n)$ changes only by an additive constant; the function $W(x, y_1, \cdots, y_n)$ is therefore determined up to an additive constant by the set of extremals (55) forming the field; we shall call it the *field-integral.*

Exactly as in § 5, *the expressions* (60) *for the partial derivatives of the field integral may be thrown into the form*

$$(79)\qquad \begin{aligned} \frac{\partial W}{\partial x} &= f(x, y_1, \cdots; p_1, \cdots) - \sum_i p_i \Omega_{n+i}(x, y_1, \cdots; p_1, \cdots; \mu_1, \cdots), \\ \frac{\partial W}{\partial y_k} &= \Omega_{n+k}(x, y_1, \cdots; p_1, \cdots; \mu_1, \cdots), \end{aligned}$$

the functions p_i, μ_ρ being defined by the equations

$$(79)\qquad \begin{aligned} p_i(x, y_1, \cdots, y_n) &= Y_i'(x; a_1, \cdots, a_n) \\ \mu_\rho(x, y_1, \cdots, y_n) &= \Lambda_\rho(x; a_1, \cdots, a_n), \end{aligned}$$

where in accordance with (64)

$$\Lambda_\rho(x, a_1, \cdots, a_n) = \Omega_\rho(x; a^0, a_1^0, \cdots, a_n^0; B_1, \cdots, B_n).$$

The equation (75) interpreted for the function $W(x, y_1, \cdots, y_n)$ contains the *extension of Kneser's fundamental theorem on transversals:*

Two transversal surfaces $\mathfrak{T}_{c'}$, $\mathfrak{T}_{c''}$ of the same field, formed by a Mayerian set of extremals, intercept on the extremals of the field arcs along which the integral J has a constant value, viz., $c'' - c'$.

Conversely: If the surface of reference for the integral $W(x, y_1, \cdots y_n)$ is the transversal surface $\mathfrak{T}_{c'}$, then the points of the field for which

$$W(x, y_1, \cdots, y_n) = c'' - c'$$

lie on the transversal surface $\mathfrak{T}_{c''}$.

Hence the transversal surfaces (77) are identieal with the surfaces

$$W(x, y_1, \cdots, y_n) = \text{const.}$$

To complete the analogy with the case $n = 1$, $r = 0$, it remains to show that the transversal surfaces can also be characterized by differential equations which are the analogue of KNESER's condition of transversality. Differentiating the identity (74) with respect to a_k and utilizing the results obtained in deriving (65), we get *

$$(80)\qquad f(x, Y_1, \cdots, Y_1', \cdots)\frac{\partial x}{\partial a_k} + \sum_i Y_i \frac{\partial Y_i}{\partial a_k}\Big|^{x=\xi} = 0.$$

* The substitution symbol $|^{x=\xi}$ refers to the whole left-hand side of equation (80).

Conversely, if ξ is a function of $a_1, \cdots, a_n$ satisfying the n partial differential equations (80), then

$$G(\xi;\ a_1, \cdots, a_n) = \text{const.},$$

i. e., independent of $a_1, \cdots, a_n$.

Hence the transversal surfaces may also be characterized by the n partial differential equations (80) for the function ξ; these differential equations are compatible in consequence of the relation (67).

If we introduce the functions η_i defined by (76), we have

$$Y'_i \frac{\partial x}{\partial a_k} + \frac{\partial Y_i}{\partial a_k}\Big|^{x=\xi} = \frac{\partial \eta_i}{\partial a_k},$$

and (80) may be written

$$(81) \qquad [f(x, Y_1, \cdots, Y'_1, \cdots) - \sum_i Y'_i V_i]\Big|^{x=\xi} \frac{\partial \xi}{\partial a_k} + \sum_i V_i\Big|^{x=\xi} \frac{\partial \eta_i}{\partial a_k} = 0.$$

On account of (24_a), (29) and (32), the same equation may also be thrown into one of the following two forms

$$(81_a) \qquad -H(x, Y_1, \cdots;\ V_1, \cdots)\Big|^{x=\xi} \frac{\partial \xi}{\partial a_k} + \sum_i V_i\Big|^{x=\xi} \frac{\partial \eta_i}{\partial a_k} = 0,$$

$$(81_b) \qquad (\Omega - \sum_i Y'_i \Omega_{n+i})\Big|^{x=\xi} \frac{\partial \xi}{\partial a_k} + \sum_i \Omega_{n+1}\Big|^{x=\xi} \frac{\partial \eta_i}{\partial a_k} = 0,$$

the arguments of Ω, Ω_{n+i} being x, $Y_1, \cdots, Y'_1, \cdots, \Lambda_1, \cdots$.

Condition (81), in any one of its three equivalent forms, is the analogue of *Kneser's condition of transversality*; for, in the simplest case $n = 1$, $r = 0$, the n equations (81_b) reduce to the one equation

$$[f(x, Y, Y') - Y' f_{y'}(x, Y, Y')]\Big|^{x=\xi} \frac{\partial \xi}{\partial a} + f_{y'}(x, Y, Y')\Big|^{x=\xi} \frac{\partial \eta}{\partial a} = 0,$$

which is the well-known condition of transversality.

§ 8. *Weierstrass' theorem for a Mayerian set of extremals.*

The fundamental formulas (78) being once established, WEIERSTRASS' theorem follows immediately by *Kneser's modification of Weierstrass' construction.**

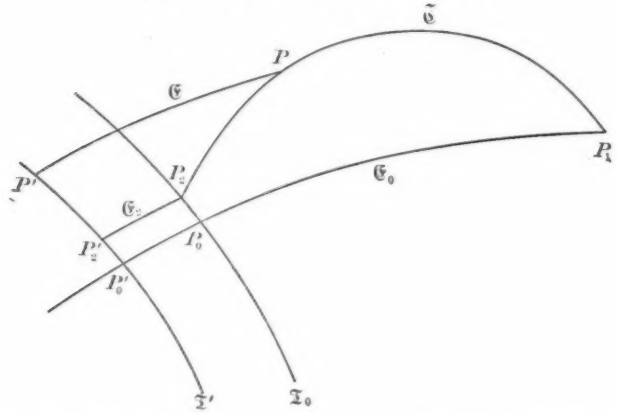
For, let $\mathfrak{T}_0$ be the transversal surface through the end-point P_0 of the extremal $\mathfrak{E}_0$, and $\mathfrak{T}'$ a second transversal surface of the field which meets the continuation of $\mathfrak{E}_0$ at a point P'_0 whose abscissa is less than x_0. From an arbitrary point P_2 of $\mathfrak{T}_0$ we draw to the second end-point P_1 of $\mathfrak{E}_0$ any curve

$$\tilde{\mathfrak{C}}: \qquad y_i = \tilde{y}_i(x)$$

*See KNESER, *Lehrbuch*, § 20, and BOLZA, *Lectures*, § 37, *a*).

of class C', lying wholly* in the field $\mathbf{S}_k$ and satisfying the differential equations

$$f_\rho(x, \tilde{y}_1(x), \cdots, \tilde{y}_1'(x), \cdots) = 0.$$



Through an arbitrary point $P(x, y_1, \cdots, y_n)$ of $\tilde{\mathfrak{C}}$ passes a unique extremal $\mathfrak{C}$ of the field, which meets the transversal surface $\mathfrak{T}'$ at one and but one point P'. We consider then the function

$$S(x) = J_{\mathfrak{C}}(P'P) + J_{\tilde{\mathfrak{C}}}(PP_1).$$

Since according to the generalized theorem on transversals,

$$J_{\mathfrak{C}_2}(P_2'P_2) = J_{\mathfrak{C}_0}(P_0'P_0),$$

(see figure), it follows exactly as in the case $n = 1$, $r = 0$, that

$$\Delta J \equiv J_{\tilde{\mathfrak{C}}}(P_2P_1) - J_{\mathfrak{C}_0}(P_0P_1) = \int_{x_2}^{x_1} S'(x)\,dx.$$

But

$$J_{\mathfrak{C}}(P'P) = W[x, \tilde{y}_1(x), \cdots, \tilde{y}_n(x)],$$

if $\mathfrak{T}'$ is used as surface of reference for W; and since the expressions for the partial derivatives of W are the same as in the special case of a set of extremals through a fixed point, also the computation of $S'(x)$ is the same as in § 5 and need therefore not be repeated here. We thus obtain WEIERSTRASS' theorem in the following form:

Let

$$y_i = Y_i(x;\, a_1, \cdots, a_n) \equiv \mathfrak{Y}_i\left(x;\, a^0, a_1, \cdots, a_n;\, \frac{\partial A}{\partial a_1}, \cdots, \frac{\partial A}{\partial a_n}\right),$$

$$\lambda_\rho = \Lambda_\rho(x;\, a_1, \cdots, a_n) \equiv \mathfrak{L}_\rho\left(x;\, a^0, a_1, \cdots, a_n;\, \frac{\partial A}{\partial a_1}, \cdots, \frac{\partial A}{\partial a_n}\right),$$

* Notice that the field $\mathbf{S}_k$ is a continuum, as follows from lemma I in § 1.

be a Mayerian n-parametric set of solutions of Euler-Lagrange's differential equations (I) *furnishing a field of extremals,* $\mathbf{S}_k$, *about the particular extremal* $\mathfrak{E}_0$. *Let further*

$$a_i = \mathfrak{a}_i(x, y_1, \cdots, y_n)$$

denote the inverse functions of the field and define

$$p_i(x, y_1, \cdots, y_n) = \Gamma_i'(x;\ \mathfrak{a}_1, \cdots, \mathfrak{a}),$$

$$\mu_\rho(x, y_1, \cdots, y_n) = \Lambda_\rho(x;\ \mathfrak{a}_1, \cdots, \mathfrak{a}_n).$$

Under these circumstances, if $\mathfrak{T}_0$ *is the transversal surface of the field passing through the end point* P_0 *of the extremal* $\mathfrak{E}_0$, P_2 *any point of* $\mathfrak{T}_0$ *and*

$$\tilde{\mathfrak{C}}: \qquad y_i = \tilde{y}_i(x), \qquad x_2 \leqq x \leqq x_1,$$

any curve of class C', *passing through the point* P_2 *and the second end-point* P_1 *of* $\mathfrak{E}_0$, *which lies wholly in the field* $\mathbf{S}_k$ *and satisfies the partial differential equations*

$$f_\rho[x, \tilde{y}_1(x), \cdots;\ \tilde{y}_1'(x), \cdots] = 0,$$

the total variation

$$\Delta J = J_{\tilde{\mathfrak{C}}}(P_2P_1) - J_{\mathfrak{C}_0}(P_0P_1)$$

is expressible by the definite integral

$$\Delta J = \int_{x_2}^{x_1} E\{x, \tilde{y}_1(x), \cdots; p_1[x, \tilde{y}_1(x), \cdots], \cdots; \tilde{y}_1'(x), \cdots; \mu_1[x, \tilde{y}(x), \cdots], \cdots\}dx,$$

where the E-function is defined by the equation

$$E(x, y_1, \cdots;\ p_1, \cdots;\ \tilde{y}_1', \cdots;\ \mu_1, \cdots) = f(x, y_1, \cdots;\ \tilde{y}_1', \cdots)$$
$$- f(x, y_1, \cdots;\ p_1, \cdots) - \sum_i (y_i' - p_i)\,\Omega_{n+i}(x, y_1, \cdots;\ p_1, \cdots;\ \mu_1, \cdots).$$

From the fundamental formulas (78) follows also immediately the extension of *Beltrami-Hilbert's independence theorem* as given by MAYER:*

If p_i, μ_ρ *are the functions of* $x, y_1, \cdots, y_n$ *defined by* (79), *then the expression*

$$T \equiv f(x, y_1, \cdots;\ p_1, \cdots) + \sum_i \left(\frac{dy_i}{dx} - p_i\right)\Omega_{n+i}(x, y_1, \cdots;\ p_1, \cdots;\ \mu_1, \cdots)$$

is a complete derivative with respect to x, *for arbitrary functions* $y_1, \cdots, y_n$ *of* x, *and at the same time*

$$f_\rho(x, y_1, \cdots;\ p_1, \cdots) = 0,$$

identically in $x, y_1, \cdots, y_n$.

*Compare MAYER II, § 3.

For, according to (78)

$$T = \frac{d}{dx} W(x, y_1, \cdots, y_n),$$

whatever functions of x may be substituted for $y_1, \cdots, y_n$, and (82) is obtained from

$$f_\rho(x, Y_1, \cdots; Y_1', \cdots) = 0$$

by substituting $\mathfrak{a}_i$ for a_i.

Hence it follows that the value of HILBERT's invariant integral generalized,* i. e., the integral

$$J_{\mathfrak{C}}^* = \int_{\mathfrak{C}} T dx,$$

taken along any admissible curve $\mathfrak{C}$ lying wholly in the field $\mathfrak{S}_k$, is independent of the path of integration and depends only on the position of the end-points, since

$$J_{\mathfrak{C}}^* = W(a'', b_1'', \cdots, b_n'') - W(a', b_1', \cdots, b_n'),$$

if $a', b_1', \cdots, b_n'$ and $a'', b_1'', \cdots, b_n''$ respectively are the coördinates of the first and second end-point of $\mathfrak{C}$.

On applying this result first to the curve $\tilde{\mathfrak{C}}$ and then to the curve $\mathfrak{E}_0$, we obtain

$$J_{\tilde{\mathfrak{C}}}^* = W(x_1, y_{11}, \cdots, y_{n1}) - W(x_2, y_{12}, \cdots, y_{n2}),$$

$$J_{\mathfrak{C}_0}^* = W(x_1, y_{11}, \cdots, y_{n1}) - W(x_0, y_{10}, \cdots, y_{n0}).$$

But since the points P_2 and P_0 lie on the same transversal surface $\mathfrak{T}_0$, we have according to § 7

$$W(x_2, y_{12}, \cdots, y_{n2}) = W(x_0, y_{10}, \cdots, y_{n0}),$$

and therefore

$$J_{\mathfrak{C}_0}^* = J_{\tilde{\mathfrak{C}}}^*.$$

On the other hand it follows from (79) that

$$p_i(x, Y_1, \cdots, Y_n) = Y'$$

and therefore in particular for $a_i = a_i^0$,

$$p_i[x, y_1(x), \cdots, y_n(x)] = y_i'(x).$$

Hence

$$J_{\mathfrak{C}_0} = J_{\mathfrak{C}_0}^* = J_{\tilde{\mathfrak{C}}}^*,$$

and therefore

$$\Delta J = J_{\tilde{\mathfrak{C}}} - J_{\tilde{\mathfrak{C}}}^*,$$

which is again WEIERSTRASS' theorem in the form given above.

FREIBURG I. B.,
April 27, 1906.

*Compare MAYER II, p. 66.

AREA OF CURVED SURFACES

BY

JAMES PIERPONT

§ 1. *Introduction.*

Let the surface S be defined by the equations

$$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v),$$

where f, g, h are continuous single valued functions in a complete limited region $\mathfrak{A}$. In case their first derivatives are continuous in $\mathfrak{A}$, i. e., in case S has a continuously turning tangent plane, the area † of S, which we may without ambiguity denote by the same letter, is

$$S = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, du\, dv, \tag{1}$$

where

$$A = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \qquad B = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}, \qquad C = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

The first attempt to show that the formula (1) still holds in case the derivatives of the coördinates are not continuous was by STOLZ. ‡ He showed the validity of (1) in the case that the six partial derivatives $\partial x/\partial u$, $\partial x/\partial v$, etc., are limited in $\mathfrak{A}$, and are continuous except at a finite number of points, or on a finite number of ordinary curves.

In the present paper I wish to show that the formula (1) holds for much wider cases.

Before doing this, it is proper to call attention to the fact that the definition of the area of a curved surface varies greatly with different authors. For example, no two of the four authors cited in the second footnote employ the same

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† HÖLDER, *Beiträge zur Potentialtheorie*, Dissertation, Stuttgart, 1882, p. 29, seq. ; JORDAN, *Cours d'Analyse*, Paris, 1893, vol. 1, p. 146, seq. ; STOLZ, *Grundzüge*, Leipzig, 1899, vol. 3, p. 239, seq. ; GOURSAT, *Cours d'Analyse*, Paris, 1902, vol. 1, p. 306, seq. ; or English translation by HEDRICK, Ginn and Co., p. 272 ; VALLÉE POUSSIN, *Cours*, vol. 2.

‡ These Transactions, vol. 3 (1902), p. 34, seq.

definition, although the four definitions lead to the same result in the case that the surface has a continuously turning tangent plane.

In seeking to remove the restriction of continuity as far as possible, I have found myself forced to adopt a type of definition similar to those employed by HÖLDER and STOLZ, which may be described briefly as follows:

Let us effect a rectangular division* of the u, v plane of norm d. By drawing the diagonals of these rectangles, we divide the plane into little right angled triangles t_κ. Such a division we shall call a *quasi rectangular division of norm* d. Corresponding to a t_κ whose vertices P_0, P_1, P_2 lie in $\mathfrak{A}$, is a plane triangle τ_κ inscribed in the surface S, and thus to a quasi rectangular division D of $\mathfrak{A}$ corresponds the inscription of a polyhedron in S whose faces are the little triangles, τ_κ. If S_D denotes the area of this polyhedron, the area of S is defined by STOLZ to be

$$\lim_{d=0} S_D. \tag{2}$$

The objection to this definition lies in the restriction of the polyhedra. One cannot help asking: would the limit of S_D be the same for a system of divisions which are not quasi rectangular? Let us therefore take a more general system of divisions. Let us divide the u, v plane into little triangles in any way, such however that no square of finite size contains an infinite number of triangles. If each triangle can be enclosed in a square of side d, we shall call such a division a *triangular division of norm* d. Corresponding to such a division of the u, v plane is a polyhedron inscribed in the surface S, whose faces are also triangular, as before. If S_D again denote the area of this polyhedron we may investigate the limit (2), and if it exists it may be called the area of S.

But when we take the divisions D in this very general manner, this limit does not exist, as SCHWARZ first pointed out.† Some restriction of D is thus necessary. HÖLDER, in the paper already referred to, showed that the limit (2) exists for all divisions D such that no angle of any triangle of D is greater than $\pi - \epsilon$, where ϵ ($\epsilon > 0$) is taken small at pleasure, and then fixed. As the sines of these angles do not sink below some positive constant, I shall call such triangular divisions *positive*.

In the following investigation I have not always been able to adopt as general a definition as does HÖLDER, who considers only the case that the first derivatives of the coördinates are continuous. On the other hand I have striven, whenever possible, to employ triangular divisions much more general than the quasi rectangular divisions of STOLZ.

* Cf. the author's *Lectures on the Theory of Functions of Real Variables*, vol. 1, Ginn and Co., 1905, § 253; or the author's papers on Multiple Integrals, these Transactions, vols. 6, 7 (1905, 1906).

† SCHWARZ, *Werke*, vol. 2, pp. 309 and 369.

§ 2. *Regular surfaces.*

Let us begin by considering a class of surfaces having at each point a tangent plane, which however does not necessarily turn continuously. Such surfaces may be called *regular.*

Let the coördinates x, y, z be one valued totally differentiable * functions in $\mathfrak{A}$. Let

$$P_0=(u_0, v_0), \qquad P_1=(u_0+\delta', v_0+\eta'), \qquad P_2=(u_0+\delta'', v_0+\eta'')$$

be the vertices of t_κ. The area of the corresponding triangle, τ_κ, is given by

$$\tau_\kappa^2=X_\kappa^2+Y_\kappa^2+Z_\kappa^2,$$

where

$$2X_\kappa=\begin{vmatrix} y_0 & z_0 & 1 \\ y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \end{vmatrix}=\begin{vmatrix} y_1-y_0 & z_1-z_0 \\ y_2-y_0 & z_2-z_0 \end{vmatrix}=\begin{vmatrix} \Delta' y & \Delta' z \\ \Delta'' y & \Delta'' z \end{vmatrix},$$

while similar expressions hold for Y_κ, Z_κ. Thus

$$S_D=\sum\sqrt{X_\kappa^2+Y_\kappa^2+Z_\kappa^2}.$$

But

$$\Delta' x=x_1-x_0=\frac{\partial x}{\partial u}\delta'+\frac{\partial x}{\partial v}\eta'+\alpha_x'\delta'+\beta_x'\eta',$$

$$\Delta'' x=x_2-x_0=\frac{\partial x}{\partial u}\delta''+\frac{\partial x}{\partial v}\eta''+\alpha_x''\delta''+\beta_x''\eta'',$$

while similar expressions hold for the other increments. Thus

$$2X_\kappa=\begin{vmatrix} \frac{\partial y}{\partial u}\delta'+\frac{\partial y}{\partial v}\eta' & \frac{\partial z}{\partial u}\delta'+\frac{\partial z}{\partial v}\eta' \\ \frac{\partial y}{\partial u}\delta''+\frac{\partial y}{\partial v}\eta'' & \frac{\partial z}{\partial u}\delta''+\frac{\partial z}{\partial v}\eta'' \end{vmatrix}+2X_\kappa',$$

where X_κ' denotes a sum of several determinants involving the infinitesimals α_y', α_y'', β_z', β_z''. Similar expressions hold for Y_κ, Z_κ. We get thus

$$X_\kappa=A_\kappa t_\kappa+X_\kappa', \qquad Y_\kappa=B_\kappa t_\kappa+Y_\kappa', \qquad Z_\kappa=C_\kappa t+Z_\kappa',$$

where A_κ, B_κ, C_κ denote the values of the determinants A, B, C at the point P_0. Thus

$$S_D=\sum t_\kappa\sqrt{\left(A_\kappa+\frac{X_\kappa'}{t_\kappa}\right)^2+\left(B_\kappa+\frac{Y_\kappa'}{t_\kappa}\right)^2+\left(C_\kappa+\frac{Z_\kappa'}{t_\kappa}\right)^2}.$$

* *Lectures*, § 423.

Let us suppose that $A^2 + B^2 + C^2$ remains greater than some positive number, while

$$\frac{X'_\kappa}{t_\kappa}, \qquad \frac{Y'_\kappa}{t_\kappa}, \qquad \frac{Z'_\kappa}{t_\kappa} \tag{3}$$

are in general uniformly evanescent in $\mathfrak{A}$; i. e., let them remain numerically less than some positive number arbitrarily small, for any division D of norm $d \leqq d_0$, except in the vicinity of a discrete set of points in $\mathfrak{A}$. Then

$$S_D = \sum t_\kappa \sqrt{A_\kappa^2 + B_\kappa^2 + C_\kappa^2} + \sum \epsilon_\kappa t_\kappa,$$

where, in general,

$$|\epsilon_\kappa| < \epsilon/\bar{\mathfrak{A}}.^*$$

If now A, B, C are limited and integrable in $\mathfrak{A}$, we have

$$\lim S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, du dv.$$

We ask now under what conditions are the expressions (3) in general uniformly evanescent? The answer is pretty evident from the example given by SCHWARZ. In fact the equation of the tangent plane at the point $x_0 y_0 z_0$ is

$$A_\kappa(x - x_0) + B_\kappa(y - y_0) + C_\kappa(z - z_0) = 0.$$

On the other hand, the plane of the triangle τ_κ is

$$\begin{vmatrix} x & y & z & 1 \\ x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = 0,$$

or

$$xX_\kappa + yY_\kappa + zZ_\kappa + U_\kappa = 0,$$

or finally

$$x\left(A_\kappa + \frac{X'_\kappa}{t_\kappa}\right) + y\left(B_\kappa + \frac{Y'_\kappa}{t_\kappa}\right) + z\left(C_\kappa + \frac{Z'_\kappa}{t_\kappa}\right) + \frac{U_\kappa}{t_\kappa} = 0.$$

Thus for (3) to converge in general uniformly to zero, it is necessary and sufficient that the planes of the faces of the inscribed polyhedra converge in general uniformly to tangent planes.

Let us call a triangular division such that the faces of the corresponding polyhedra converge in general uniformly to tangent planes a *regular division*. We have now the following:

THEOREM 1. *Let the coördinates x, y, z be one valued totally differentiable functions of the parameters u, v in the limited complete region $\mathfrak{A}$, such that*

* $\bar{\mathfrak{A}} = \overline{\text{Cont}}\ \mathfrak{A}$ denotes upper content of $\mathfrak{A}$. Cf. *Lectures*, § 702.

$A^2 + B^2 + C^2$ is greater than some positive constant, and is limited and integrable in $\mathfrak{A}$. Then

$$S = \lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, du\, dv,$$

D denoting a regular triangular division of norm d.

If we call this limit *area*, we have at once a number of its properties. We mention only the following:

THEOREM 2. *Let $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ be unmixed* partial complete regions of $\mathfrak{A}$, whose union is $\mathfrak{A}$. Let $S_1, \cdots, S_m$ be the pieces of S corresponding to them. Then each S_κ has an area, and their sum is S.*

THEOREM 3. *Let $\mathfrak{A}_\lambda$ be an unmixed partial complete region of $\mathfrak{A}$ depending on a parameter $\lambda \doteq 0$, such that $\lim \bar{\mathfrak{A}}_\lambda = \bar{\mathfrak{A}}$. Then*

$$\lim_{\lambda=0} S_\lambda = S.$$

THEOREM 4. *The area of S remains unaltered when S is subjected to a displacement or when the parameters u, v are subjected to the transformation of theorems 31, 32 in the author's paper† on Improper Integrals.*

§ 3. *Irregular surfaces.*

We turn now to certain surfaces which do not have tangent planes at every point, i. e., surfaces for which one or more of the first partial differential coefficients of the coördinates do not exist at certain points, and which may be styled *irregular surfaces.*

THEOREM 5. *Let the coördinates x, y, z be one valued functions, having limited total difference quotients‡ in the limited complete region $\mathfrak{A}$. Let D be a positive division of norm $\leqq$ some number d_0. Then*

$$\max S_D$$

is finite, and evanescent with $\bar{\mathfrak{A}}$.

For, let the difference quotients remain $\leqq \mu$. We have

But $$S_D \leqq \Sigma |X_\kappa| + \Sigma |Y_\kappa| + \Sigma |Z_\kappa|.$$

$$|X_\kappa| = \tfrac{1}{2} |\Delta' y \Delta'' z - \Delta' z \Delta'' y| \leqq \tfrac{1}{2} \{ |\Delta' y| \cdot |\Delta'' z| + |\Delta' z| \cdot |\Delta'' y| \}$$

$$\leqq \mu^2 \overline{P_0 P_1} \cdot \overline{P_0 P_2} = \mu^2 t_\kappa |\csc \theta_\kappa|;$$

where θ_κ is the angle made by the sides $\overline{P_0 P_1}$, $\overline{P_0 P_2}$. As D is a positive

* *Lectures*, § 711.

† These Transactions, vol. 7 (1906), p. 172.

‡ *Lectures*, § 708.

division, one of the angles of t_κ is such that $\csc \theta_\kappa$ is numerically less than some positive number M. Thus

$$|X_\kappa| < \mu^2 M t_\kappa,$$

where μ, M are independent of κ and d. Similar relations hold for $|Y_\kappa|$, $|Z_\kappa|$; and the truth of the theorem is now obvious.

THEOREM 6. *In the limited complete region $\mathfrak{A}$, let the coördinates x, y, z be as in theorem* 1, *except that at certain points forming a discrete aggregate $\mathfrak{a}$, the first partial differential coefficients do not exist. Let their total difference quotients be limited in $\mathfrak{A}$. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, du\, dv,$$

where D denotes a positive regular division of norm d.

Let us first show the limit on the left exists. We may choose an unmixed partial region $\mathfrak{B}$ of $\mathfrak{A}$, containing $\mathfrak{a}$ as inner* points, such that $\overline{\mathfrak{B}}$ is as small as we please. Let $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$, while S_C may denote the area of the surface corresponding to $\mathfrak{C}$. The triangles t_κ fall into two groups; first a group G_1 containing points of $\mathfrak{B}$, and second a group G_2, containing only points of $\overline{\mathfrak{B}}$. Then

$$S_D = \sum \sqrt{X_\kappa^2 + Y_\kappa^2 + Z_\kappa^2} = \sum_{G_1} + \sum_{G_2}.$$

But $\overline{\mathfrak{B}}$ may be chosen so small that the first sum is $< \epsilon/4$ for any $d < d_0$. Moreover, by taking d_0 still smaller if necessary, we have

$$|\textstyle\sum_{G_2} - S_C| < \epsilon/4.$$

Hence

$$|S_D - S_C| < \epsilon/2, \qquad (d < d_0). \tag{4}$$

Similarly for any other division D' of norm d',

$$|S_{D'} - S_C| < \epsilon/2, \qquad (d' < d_0),$$

decreasing d_0 still farther if necessary. Thus

$$|S_{D'} - S_D| < \epsilon, \qquad (d, d' < d_0).$$

Hence $\lim S_D$ exists; call it S. Since S exists, we may take d_0 so small that

$$|S - S_D| < \epsilon/2, \qquad (d < d_0).$$

* *Lectures*, § 704, 4.

This with (4) gives

$$|S - S_C| < \epsilon;$$

that is,

$$S = \lim S_C = \lim \int_{\mathfrak{A}} = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, du dv,$$

by *Lectures*, § 724.

The preceding theorem takes care of a large class of irregular surfaces whose total difference quotients are limited, either with the original set of parameters or with a new set. When the total difference quotients are not limited, and it is not practical to transform to a new system, whose total difference quotients are limited, the following theorems may be useful. They rest on a new system of divisions which we define as follows. Let us effect a quadrate division of the u, v plane of norm d, and take the triangles t_κ so that for any triangular division D associated with d, no square contains more than n triangles, and no triangle lies in more than ν squares; n, ν being arbitrarily large constants independent of d. Such a division we call a *quasi quadrate division of norm* d.

We shall also need to introduce a new classification of functions according to their variation in $\mathfrak{A}$, or along lines parallel to the u, v axes.

Let D be a quadrate division of the u, v plane, of norm $d \leqq d_0$. Let $\omega_\kappa = \operatorname{osc} \phi(u, v)$ denote the oscillation of ϕ in one of the squares of D. The extreme

$$\max_D \sum \omega_\kappa d,$$

for all divisions D is called* the variation of ϕ in $\mathfrak{A}$. If the variation is not only limited, but evanescent with $\bar{\mathfrak{A}}$, we shall say ϕ has *limited fluctuations in* $\mathfrak{A}$. Let $\mathfrak{a}$ denote the points of $\mathfrak{A}$ lying on a line parallel to the u axis. Let us effect a division D' of this line into intervals of length $\leqq d$. Let $\omega'_\kappa = \operatorname{osc} \phi(u, v)$ in one of these intervals. Then

$$\tilde{\omega}_{\mathfrak{a}} = \max_{D'} \sum \omega'_\kappa$$

denotes the variation of ϕ in $\mathfrak{a}$.

Let us consider now *all* the aggregates $\mathfrak{a}$ lying on lines parallel to the u axis, and suppose $\bar{\mathfrak{a}} \leqq \sigma$, where $\sigma \doteq 0$. If now for each there exists a constant G, such that $\tilde{\omega}_{\mathfrak{a}} < \sigma G$; that is, if $\tilde{\omega}_{\mathfrak{a}}$ are uniformly evanescent with σ, we shall say that $\phi(u, v)$ has *limited fluctuation with respect to* u. Similar remarks apply to v.

With the aid of these notions we may prove the following theorems.

THEOREM 7. *Let the coördinates x, y, z be one valued limited functions in the limited complete region $\mathfrak{A}$. Let x, y have limited total difference quotients,*

* *Lectures*, § 709.

while z has limited variation in $\mathfrak{A}$. Let D denote a quasi quadrate division of norm $d \leqq d_0$. Then

$$\max_D S_D$$

is finite.

For, as before

$$2|X_\kappa| \leqq |\Delta'_y| \cdot |\Delta''_z| + |\Delta''_y| \cdot |\Delta'_z|.$$

But μ denoting a sufficiently large constant,

$$|\Delta'_y|,\ |\Delta''_y| < \mu d.$$

Let $\omega_\iota = \operatorname{osc} z$ in the square s_ι. If the triangle t_κ lies in the squares $s_{\iota_1}, \cdots, s_{\iota_\nu}$,

$$|\Delta'_z|,\ |\Delta''_z| \leqq \omega_{\iota_1} + \cdots + \omega_{\iota_\nu}.$$

Hence

$$|\Delta'_z| + |\Delta''_z| \leqq 2(\omega_{\iota_1} + \cdots + \omega_{\iota_\nu}).$$

Thus, n, ν, M, M' denoting sufficiently large constants

$$\Sigma|X_\kappa| < \mu n \Sigma d(\omega_{\iota_1} + \cdots + \omega_{\iota_\nu}),$$

the summation extending over all squares s_ι containing points of $\mathfrak{A}$. But z having limited variation

$$\Sigma \omega_\iota d < M. \tag{5}$$

Hence

$$\Sigma|X_\kappa|,\ \Sigma|Y_\kappa| < \mu n \nu M.$$

Finally as in theorem 5,

$$\Sigma|Z_\kappa| < M'.$$

The theorem is thus established.

THEOREM 8. *The coördinates x, y, z being as in the preceding theorem, except that z has limited fluctuation in $\mathfrak{A}$, and D denoting a quasi quadrate division of norm $d \leqq d_0$,*

$$\max_D S_D$$

is finite, and evanescent with $\bar{\mathfrak{A}}$.

The reasoning is the same as in theorem 8, except that now M and M' are evanescent with $\bar{\mathfrak{A}}$.

THEOREM 9. *Let the coördinates x, y, z have limited total difference quotients in $\mathfrak{A}$, while the variation of z along any line parallel to the u or v axes is $< M$. Let $\mathfrak{A}$ lie in a square of side $s \doteq 0$. Then*

$$\max S_D < Gs;$$

where G is some constant independent of s, and D is a quasi rectangular division of norm $\leqq d_0$.

For, here

$$2\sum |X_\kappa| \leqq \sum |\Delta' y| \cdot |\Delta'' z| + \sum |\Delta'' y| \cdot |\Delta' z| < M' \sum \omega_u d_v + M' \sum \omega_v d_u ;$$

where M' denotes a sufficiently large constant; d_u, d_v denote the length of sides of one of the triangles t_κ parallel respectively to the u, v axes, and ω_u, ω_v the oscillation of z along these sides. Since the variation is $< M$ in both directions,

$$\sum \omega_u d_v = \sum_v d_v \sum \omega_u < M \sum d_v < Ms .$$

Similarly,

$$\sum \omega_v d_u < Ms .$$

The rest of the proof follows as before.

THEOREM 10. *The symbols having the same meaning as before, except that z has limited fluctuation with respect to u, v,*

$$\operatorname*{Max}_D S_D < Gs^2 .$$

The demonstration is similar to the foregoing.

Following the line of proof used in establishing theorem 6, and employing the theorems just given, we readily prove the following theorems:

THEOREM 11. *Let $\mathfrak{A}$ be a limited complete region, containing the discrete aggregate $\mathfrak{a}$. Let $\mathfrak{b}$ be a partial region of $\mathfrak{A}$ containing the points $\mathfrak{a}$, such that $\mathfrak{B} = \mathfrak{A} - \mathfrak{b}$ is exterior to $\mathfrak{a}$, moreover let $\bar{\mathfrak{b}} \doteq 0$. Let the coördinates x, y, z be one valued totally differentiable functions in $\mathfrak{B}$, such that $A^2 + B^2 + C^2$ never sinks below a positive constant in any $\mathfrak{B}$, is properly integrable in any $\mathfrak{B}$, and improperly integrable in $\mathfrak{A}$. Let x, y have limited total difference quotients and z limited fluctuation in $\mathfrak{b}$. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, du dv ,$$

where D is any quasi quadrate division of norm d.

THEOREM 12. *Let the symbols have the same meaning as in the preceding theorem except that:* 1) *$\mathfrak{a}$ reduces to a finite aggregate;* 2) *z has limited variation along any line parallel to the u, v axes;* 3) *D denotes a regular quasi rectangular division. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} .$$

THEOREM 13. *Let the symbols have the same meaning as in theorem* 11 *except that:* 1) z *has limited fluctuation with respect to* u, v *in* $\mathfrak{b}$; 2) D *denotes a regular quasi rectangular division. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{F}}.$$

THEOREM 14. *If we call the limits in theorems* 11, 12, 13 *area, the theorems* 2, 3, 4 *still hold.*

ROCK BOUND, SACHEM'S HEAD,
April, 1906.

ON MULTIPLY TRANSITIVE GROUPS*

BY

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The properties of primitive groups that contain transitive subgroups of lower degree were first investigated by JORDAN.† In the *Traité des Substitutions* he proved that if a primitive group of degree n contains a circular substitution of prime degree (p) it is at least $n-p+1$ times transitive. The capital importance of this theorem led him to examine the general and much more difficult case, that in which the subgroup of lower degree is merely transitive. He obtained the remarkable theorem:‡

"If a primitive group G of degree n contains a group Γ, the substitutions of which displace only p letters and permute them transitively (p being any integer), it is at least $n-p-2q+3$ times transitive, q being the greatest divisor of p such that we can arrange the letters of Γ in two different ways in systems of q letters which have the property that each substitution of Γ replaces the letters of each system by those of a single system. If none of the divisors of p have this property (which will happen notably if Γ is primitive, or formed of the powers of the same circular substitution) G is $n-p+1$ times transitive."

NETTO § and RUDIO ‖ later gave proofs for that special case in which Γ is primitive. The only other contribution to this theory was made by MARGGRAFF.¶ He proved that if q is the greatest divisor of p such that the letters of Γ may be arranged in systems of imprimitivity in at least *three* different ways, and if p is divisible by some number r such that Γ admits $r+1$ systems of imprimitivity with one letter in common and no two of which have more than one letter in common, G is at least $n-p-2q+3$ times transitive. If these two conditions are not fulfilled, G is $n-p+1$ times transitive. In MARGGRAFF's

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† C. JORDAN, *Traité des Substitutions*, 1870, p. 664.

‡ C. JORDAN, Journal de Mathématiques, ser. 2, vol. 16 (1871), pp. 383–408.

§ NETTO, Journal für Mathematik, vol. 83 (1877), pp. 43–56.

‖ RUDIO, Journal für Mathematik, vol. 102 (1888), pp. 1–9.

¶ MARGGRAFF, Dissertation, *Ueber primitive Gruppen mit transitiven Untergruppen geringeren Grades*, Giessen, 1889; and also, *Wissenschaftliche Beilage zum Jahresberichte des Sophien-Gymnasiums zu Berlin*, 1895, Programm nr. 65.

second paper are given some interesting relations between n and p, for example, $p \geqq \frac{1}{2}n$.

The word "two" in JORDAN's theorem, and "three" in MARGGRAFF's may in all cases be replaced by "$q+1$."

Let it be assumed that G is a doubly (and not triply) transitive group of degree $\sum_{x=1}^{r+1} q_x$ $(q_{r+1}=1)$, in which are found transitive subgroups H^i $(i=1, 2, \cdots, r)$ of degrees $\sum_{x=1}^{i} q_x$ respectively. It is also assumed that the degree of any transitive subgroup of G which displaces more than q_1 and less than $\sum_{x=1}^{r} q_x$ letters is one of the numbers $\sum_{x=1}^{i} q_x$ $(i=2, 3, \cdots, r-1)$. This does not say that there is not a transitive subgroup of degree less than q_1. Let $a_{1,k}^i$ $(k=1, 2, \cdots, q_i)$ be the letters displaced by H^i but not by H^{i-1}, and let Γ_1^i be the largest subgroup of G on the letters of H^i; Γ_1^i includes all the preceding subgroups and is transitive.

If $q_r=1$, G is triply transitive contrary to hypothesis: hence $q_r>1$. If for those values of i corresponding to which $q_i>1$, the letters $a_{1,k}^i$ $(k=1, 2, \cdots, q_i)$ do not form a system of imprimitivity in Γ_1^i, some substitution S of Γ_1^i will give a group $S^{-1}\Gamma_1^{i-1}S$ displacing some of the letters $a_{1,k}^i$ $(k=1, 2, \cdots, q_i)$ and leaving fixed at least one of them. Then $\{\Gamma_1^{i-1}, S^{-1}\Gamma_1^{i-1}S\}$ is a transitive group of degree greater than $\sum_{x=1}^{i-1} q_x$ and less than $\sum_{x=1}^{i} q_x$, contrary to hypothesis. By the definition of imprimitivity all the subgroups of Γ_1^i, notably $\Gamma_1^1, \Gamma_1^2, \cdots, \Gamma_1^{i-1}$, admit these systems. It should also be remarked that the letters $a_{1,k}^x$ $(k=1, 2, \cdots, q_x)$ fall into sets of imprimitivity q_i by q_i, for $x=1, 2, \cdots, i-1$. Since $q_r>1$, then $q_i>1$ $(i=1, 2, \cdots, r-1)$.

Let $a_{2,1}^r$ represent the system $a_{1,k}^r$ $(k=1, 2, \cdots, q_r)$ of Γ_1^r, and let $a_{2,k}^i$ $(i=1, 2, \cdots, r-1;\ k=1, 2, \cdots, q_i/q_r)$ be the remaining systems into which $a_{2,1}^r$ goes under the substitutions of Γ_1^r. We define Γ_2^i as the group in the systems $a_{2,k}^x$ $(x=1, 2, \cdots, i;\ k=1, 2, \cdots, q_x/q_r)$ of Γ_1^i. Obviously Γ_2^r is at least doubly transitive.

The group G, being doubly transitive, contains a substitution $T_{i_1}=(a_{1,1}^{r-1}a_{1,i_1}^r)\cdots$. Let it be granted for the moment that T_{i_1} transforms $\Gamma_1^{i+1}, \Gamma_1^{i+2}, \cdots, \Gamma_1^{r-1}$ each into itself and leaves fixed the letters $a_{1,1}^{i+1}, a_{1,1}^{i+2}, \cdots, a_{1,1}^{r-1}$. Then $\{T_{i_1}^{-1}\Gamma_1^i T_{i_1}, \Gamma_1^i\}_{j}=\Gamma_1^i$; for, its degree is less than $\sum_{x=1}^{i+1} q_x$ and must consequently be $\sum_{x=1}^{i} q_x$, that of Γ_1^i. If T_{i_1} displaces $a_{1,1}^i$ there is a substitution S in Γ_1^i which replaces $a_{1,1}^i$ by the same letter as does T_{i_1}. Then $S^{-1}T_{i_1}$ leaves fixed $a_{1,1}^i$ as well as $a_{1,1}^{i-1}, \cdots, a_{1,1}^r$ and may be used for T_{i_1}. But it is clear that $T_{i_1}=(a_{1,1}^{r+1}a_{1,i_1}^r)\cdots$ transforms Γ_1^{r-1} into itself, and by multiplication by a substitution of Γ_1^{r-1} a like substitution may be obtained which leaves $a_{1,1}^{r-1}$ fixed. Then by a complete induction G contains a substitution

$$T_{i_1}=(a_{1,1}^1)(a_{1,1}^2)\cdots(a_{1,1}^{r-1})(a_{1,1}^{r+1}a_{1,i_1}^r)\cdots$$

for $i_1=1, 2, \cdots, q_r$, which transforms each group $\Gamma_1^1, \Gamma_1^2, \cdots, \Gamma_1^{r-1}$ into itself.

Each system of letters $a^i_{1,k}$ $(k = 1, 2, \cdots, q_i)$ is transformed into itself. The symbol $a^{r-1}_{3,1}$ will be introduced later for the system $a^{r-1}_{1,k}$ $(k = 1, 2, \cdots, q_{r-1})$. and other properties of T_{i_1} pointed out.

Now $T^{-1}_{i_1}\Gamma^{r-1}_1 T_{i_1} = \Gamma^{r-1}_1$ $(i_1 = 1, 2, \cdots, q_r)$, but the system $a^{r-1}_{2,1}$ is replaced by q_r other systems which have at least the letter $a^{r-1}_{1,1}$ in common with $a^{r-1}_{2,1}$. Let these q_r new systems of imprimitivity of Γ^{r-1}_1 be represented by $a^{r-1}_{2,1,i_1}$. If it is assumed that $a^{r-1}_{2,1,i_1}$ has a second letter $a^{r-1}_{1,2}$ in common with $a^{r-1}_{2,1}$, T_{i_1} has the sequence $a^{r-1}_{1,k}\, a^{r-1}_{1,2}$, where $a^{r-1}_{1,k}$ is some letter of the system $a^{r-1}_{2,1}$. In Γ^r_2 there is a substitution $S = (a^{r-1}_{2,1}\, a^r_{2,1})\cdots$. Now $U = S^{-1}T_{i_1}S$ leaves fixed the letter of $a^{r-1}_{2,1}$ which follows $a^{r-1}_{1,1}$ in S, replaces the letter which follows $a^{r-1}_{1,k}$ in S by the one which follows $a^{r-1}_{1,2}$, and certainly replaces a letter of $a^{r-1}_{2,1}$ by $a^{r+1}_{1,1}$. Hence one sees that the degree of $\{U^{-1}\Gamma^{r-1}_1 U, \Gamma^{r-1}_1\}$ is greater than $\sum_{x=1}^{r-1} q_x$ and less than $\sum_{x=1}^{r} q_x$, contrary to hypothesis. *The q_r systems $a^{r-1}_{2,1,i_1}$ have one and only one letter in common with $a^{r-1}_{2,1}$.*

Suppose it possible to find in G a substitution T that transforms Γ^{r-1}_1 into itself, and which replaces the system $a^{r-1}_{2,1,i_1}$ by a system α' distinct from any system resulting from the $q_r + 1$ arrangements obtained above. If α' does not include the letter $a^{r-1}_{1,1}$, some substitution S'' of Γ^{r-1}_1 will replace α' by a system α which does include $a^{r-1}_{1,1}$. Now $T_{i_1}TS''$ replaces the system $a^{r-1}_{2,1}$ by α and the letter $a^{r-1}_{1,1}$ by $a^{r-1}_{1,k}$ say. We take from Γ^{r-1}_1 a substitution S' which replaces $a^{r-1}_{1,k}$ by $a^{r-1}_{1,1}$, and in consequence leaves α fixed; the product $U = T_{i_1}TS''S'$ leaves $a^{r-1}_{1,1}$ fixed and replaces $a^{r-1}_{2,1}$ by α, so that $U = (a^{r-1}_{1,1})(a^{r+1}_{1,1}\, a^r_{1,k}\cdots)\cdots$. Now $UT^{-1}_k = (a^{r-1}_{1,1})(a^{r+1}_{1,1})\cdots = S$, a substitution of Γ^r_1, and therefore $U = ST_k$, $k \leqq q_r$. Hence α, the result of transforming Γ^{r-1}_1 first by S and then by T_k, is merely one of the systems $a^{r-1}_{2,1,i_1}$.

The system $a^{r-1}_{2,1,i_1}$ bears the same relation to Γ^{r-1}_1, $T^{-1}_{i_1}\Gamma^r_1 T_{i_1}$, and G, that $a^{r-1}_{2,1}$ does to Γ^{r-1}_1, Γ^r_1, and G. From $a^{r-1}_{2,1,i_1}$ then can be obtained, by means of substitutions that leave $a^{r-1}_{1,1}$ fixed and Γ^{r-1}_1 invariant, q_r other systems with but one letter in common with $a^{r-1}_{2,1}$. But we have just seen that these q_r systems will coincide with systems already obtained. Hence *the letters of Γ^{r-1}_1 may be arranged in systems of imprimitivity of q_r letters each in at least $q_r + 1$ ways with one letter common to the $q_r + 1$ systems and with no other letter common to any two of them.* This theorem holds *à fortiori* for all transitive subgroups of Γ^{r-1}_1, in particular for H^1. It was first proved by MARGGRAFF, loc. cit.

Since T_{i_1} permutes the letters $a^{r-1}_{1,k}$ $(k = 1, 2, \cdots, q_{r-1})$ among themselves, *the letters (q^2_r in number) involved in the systems $a^{r-1}_{2,1}$, $a^{r-1}_{2,1,1}$, $a^{r-1}_{2,1,2}$, $\cdots$, $a^{r-1}_{2,1,q_r}$, are all found in the larger system $a^{r-1}_{1,k}$ above. Hence we have the important relation $q_{r-1} \geqq q^2_r$.*

It will now be shown that Γ^r_2 is not in general triply transitive. An exception arises only when $r = 2$. Suppose Γ^r_2 triply transitive. To each of the $q_r + 1$ distinct arrangements of the letters of Γ^{r-1}_1 in systems of imprimitivity of

q_r letters each, corresponds a doubly transitive group according to which systems are permuted.* Let $\Gamma_1^{r-1}(a_{1,1}^{r-1})$ be the subgroup of Γ_1^{r-1} that leaves fixed the letter $a_{1,1}^{r-1}$. The remaining $q_r - 1$ letters of each system of imprimitivity to which $a_{1,1}^{r-1}$ belongs may be permuted only among themselves by the substitutions of $\Gamma_1^{r-1}(a_{1,1}^{r-1})$. Now $\Gamma_2^{r-1}(a_{2,1}^{r-1})$ is transitive of degree $(1/q_r)\sum_{x=1}^{r-1} q_x - 1$, while the number of systems transitively connected by it cannot exceed $q_r - 1$. Hence $\sum_{x=1}^{r-1} q_x \leqq q_r^2$, and (since $q_{r-1} \geqq q_r^2$), $q_{r-1} = q_r^2$ and $r = 2$, unless Γ_1^{r-1} is a regular group, in which case also $r = 2$. It will be shown later that an imprimitive group of degree q^2 in which $q + 1$ systems of q letters each with one letter in common, and such that no two of these systems have more than one letter in common, is either regular or of class $q^2 - 1$. However, the proof of the theorem we are going to establish is already complete for $r = 2$, whether Γ_2^r is triply transitive or not. In completing the proof we assume then that $r > 2$, so that Γ_2^r *is doubly but not triply transitive.* It is such a group as G itself and to it may be applied all preceding results for G.

Is it possible for Γ_2^r to have a transitive subgroup of degree greater than q_1/q_r and not equal to one of the numbers $(1/q_r)\sum_{x=1}^{i} q_x (i = 2, 3, \cdots, r)$? Let, if possible, $\Gamma_2^{i,k}$ be a transitive subgroup of Γ_2^{i+1} which displaces besides the letters of Γ_2^i the k letters $a_{2,1}^{i+1}, a_{2,2}^{i+1}, \cdots, a_{2,k}^{i+1} (1 \leqq k < q_{i+1}/q_r)$. Then between Γ_1^i and Γ_1^{i+1} there is a corresponding subgroup $\Gamma_1^{i,k}$, which includes Γ_1^i and is transitive in the systems $a_{2,1}^1, \cdots$, so that $\Gamma_1^{i,k}$ certainly has a transitive constituent of degree $\sum_{x=1}^{i} q_x + kq_r$. Now transform Γ_1^i by all the substitutions of $\Gamma_1^{i,k}$. If the group generated by Γ_1^i and all these transforms is not transitive of degree $\sum_{x=1}^{i} q_x + kq_r$, it is because $kq_r \geqq \sum_{x=i}^{i} q_x$, an absurdity since $kq_r < q_{i+1}$, $q_{i+1} \leqq q_i$, that is, $kq_r < q_i$. This holds for $i = 1, 2, \cdots, r-2$. Nothing is said about possible transitive subgroups of various degrees in Γ_2^1.

We may now form the doubly transitive group in the systems of Γ_2^{r-1}. The system $a_{2,k}^{r-1}$ $(k = 1, 2, \cdots, q_{r-1}/q_r)$ will be represented by $a_{3,1}^{r-1}$. The large system $a_{2,k}^i$ $(k = 1, 2, \cdots, q_i/q_r)$ breaks up into the smaller systems $a_{3,k}^i$ $(k = 1, 2, \cdots, q_i/q_{r-1})$. This group on the letters $a_{3,k}^i$ $(i = 1, 2, \cdots, r-1)$ will be indicated by Γ_3^{r-1}, and the subgroup of it which corresponds to Γ_2^i is Γ_3^i. In the same way we proceed to form the successive groups Γ_j^i in the sys-

*Cf. C. JORDAN, Journal de Mathématiques, ser. 2, vol. 16 (1871), third paragraph of article 7 on page 388. From the fact that "I" is doubly transitive with respect to the systems of q letters $s, s_1, \cdots$ which it contains, it does not follow that if its letters can be grouped in different ways in systems of imprimitivity, each of the new systems will be contained completely within one of the systems $s, s_1, \cdots$. For example it is not true in the regular non-cyclic group of order 6. In it systems of two letters each are permuted according to a doubly transitive group, while there are three distinct ways of arranging the letters of the group in systems of imprimitivity of two letters each. But the argument is valid if use is made of the fact that "I" contains substitutions leaving fixed at least two letters. To see how the subgroup "H" of "I" should be brought in to complete the proof, see MARGGRAFF, l. c., theorem X (1889), p. 20, and (1895), p. 16.

tems $a^i_{j,k}$ $(k=1,2,\cdots,q_i/q_{r-j+2};\ i,j=1,2,\cdots,r$, provided $i+j\leqq r+2$; and $q_{r+1}=1)$. The groups Γ^i_j for which $i+j=r+2$ are all doubly and not triply transitive except perhaps Γ^2_r. *From* Γ^{i+1}_{r-i+1} *we have*

$$\frac{q_{i-1}}{q_{i+1}}\geqq\left(\frac{q_i}{q_{i+1}}\right)^2,\qquad or\qquad q_{i-1}\geqq\frac{q_i^2}{q_{i+1}}\qquad(i=2,3,\cdots,r),$$

so that

$$q_i\geqq q_{i+1}\sqrt[r-i]{q_{i+1}},\qquad or\qquad q_i\geqq q_r^{r-i+1},\qquad or\qquad q_i\leqq q_1^{(r-i+1)/r}.$$

The operator $T_{i_1}=(a^{r+1}_{1,1}a^r_{1,i_1})\cdots$, we have seen, leaves fixed the letters $a^1_{1,1}$, $a^2_{1,1},\cdots,a^{r-1}_{1,1}$, and the systems $a^{r-1}_{3,1}$, $a^{r-2}_{4,1}$, $a^{r-3}_{5,1},\cdots,a^2_{r,1}$, $a^1_{r+1,1}$. Since T_{i_1} leaves fixed the system $a^{r-j+2}_{j,1}$ of Γ^{r-j+2}_j $(j=3,4,\cdots,r)$, it permutes the systems $a^i_{j,k}$ $(k=1,2,\cdots,q_i/q_{r-j+2};\ i=1,2,\cdots,r-j+2)$ among themselves, without a change to an arrangement essentially distinct. Then T_{i_1} leaves fixed all the systems $a^i_{j,1}$ $(j=1,2,\cdots,r+1;\ i=1,2,\cdots,r-1$, provided $i+j\leqq r+2$, and, when $j=2$, $i\neq r-1)$. In exactly the same way we get a substitution $T_{i_j}=(a^{r-j+2}_{j,1}a^{r-j+1}_{j,i_j})\cdots$ which leaves fixed all those elements which bear to Γ^{r-j+2}_j the same relation as the elements left fixed by T_{i_1} bear to G. If we consider the group Γ^r_y it is clear that a substitution S can always be found in it that replaces a certain letter $a^r_{y,z}$ by $a^r_{y,1}$ and that has the property of leaving fixed each of the letters $a^1_{y,1}$, $a^2_{y,1}$, $a^{r-1}_{y,1}$, and hence also the systems $a^1_{y+1,1}$, $a^2_{y+1,1},\cdots,a^{r-1}_{y+1,1}$, $a^1_{y+2,1},\cdots$. With the aid of such substitutions we may choose T_{i_j} so that it leaves fixed all the elements $a^i_{y,1}$ $(y=1,2,\cdots,r+1;\ i=1,2,\cdots,r-j$, provided $y+i\leqq r+2$, and $i\neq r-j$ when $y=j)$. It is clear that T_{i_j} transforms Γ^i_1 $(i=1,2,\cdots,r-j)$ and Γ^{r-j+2}_1 each into itself. We should bear in mind that T_{i_j} may also be regarded as a substitution of Γ^{r-j+2}_1 and leaves fixed all the letters $a^i_{1,k}$ $(k=1,2,\cdots,q_i;\ i=r-j+3,\cdots,r+1)$.

The following theorem will now be proved by a complete induction:

The letters of Γ^{i-1}_1 *may be grouped* q_i *by* q_i *in systems of imprimitivity which have an arbitrary letter* $a^{i-1}_{1,1}$ *in common in at least* $q_i\sum_{x=i}^{r+1}1/q_x$ *distinct ways. Furthermore,* (1) *Any two of these systems have exactly* q_{i+1} *letters in common.* (2) *The letters involved in these systems are all included in the larger system* $a^{i-1}_{r-i+3,1}$. (3) *No substitution of* G *which leaves* Γ^{i-1}_1 *invariant can replace one of these systems by a system of imprimitivity not included among those into which one of them goes by a substitution of* Γ^{i-1}_1 *itself.* (4) *Those systems which have* $a^{i-1}_{2,1}$ *in common preserve the original systems* $a^i_{2,k}$. (5) *The* q_i *systems of* q_i *letters which have* $a^{i-1}_{1,1}$ *but not* $a^{i-1}_{2,1}$ *in common contain no letter of the system* $a^{i-1}_{2,1}$ *except* $a^{i-1}_{1,1}$.

We assume the truth of the above theorem for the subgroup Γ^i_1 and its systems of q_{i+1} letters, and shall show that it must then hold as stated for Γ^{i-1}_1. Now if it holds for Γ^i_1 it holds also for Γ^{i-1}_2. Hence systems of q_i/q_r letters

with $a_{2,1}^{i-1}$ in common may be chosen in $q_i \sum_{x=i}^{r} 1/q_x$ distinct ways, $q_i \sum_{x=i}^{r-1} 1/q_x$ of which have the system $a_{3,1}^{i-1}$ in common. Now transform Γ_1^{i-1} by T_{i_1} $(i_1 = 1, 2, \cdots, q_r)$. Since T_{i_1} does not replace the systems $a_{3,k}^{i}$ of Γ_2^{i-1} by other systems essentially distinct, the $q_i \sum_{x=i}^{r-1} 1/q_x$ systems with $a_{3,1}^{i-1}$ in common undergo no change. But since T_{i_1} leaves fixed $a_{3,1}^{i-1}$ and may replace letters of $a_{2,1}^{i-1}$ only by letters of $a_{3,1}^{i-1}$, and since $a_{1,1}^{i-1}$ is the only letter of $a_{3,1}^{i-1}$ in the remaining q_i/q_r systems, no two of the block of $(q_i/q_r)(1 + q_r)$ systems obtained by this transformation are the same. Since T_{i_1} leaves fixed the system $a_{r-i+2,1}^{i-1}$ (the system of q_i letters from which we start), each system of the total number, $q_i \sum_{x=i}^{r-1} 1/q_x + q_i(1 + q_r)/q_r = q_i \sum_{x=i}^{r+1} 1/q_x$, has just q_{i+1} letters in common with $a_{r-i+2,1}^{i-1}$. Obviously all the letters of all these systems are included in $a_{r-i+3,1}^{i-1}$. The next point to establish is that no substitution of G under which Γ_1^{i-1} is invariant can replace any of the $q_i \sum_{x=i}^{r+1} 1/q_x$ systems with $a_{1,1}^{i-1}$ in common (the system a say) by a system (α'') not found among the $\sum_{x=1}^{i-1} q_x \cdot \sum_{x=i}^{r+1} 1/q_x$ systems of Γ_1^{i-1} resulting from the above systems which have $a_{1,1}^{i-1}$ in common, after transformation by all the substitutions of Γ_1^{i-1}. By hypothesis all substitutions of Γ_2^{r} under which Γ_2^{i-1} is invariant give rise to no new systems. Let T be a substitution of G which replaces the system a by α'', and let $T^{-1}\Gamma_1^{i-1}T = \Gamma_1^{i-1}$. Let T_{i_1} be that substitution (defined as before) which replaces one of the systems of q_i letters which include $a_{2,1}^{i-1}$, by a. Since $T_{i_1}T$ cannot be a substitution of Γ_2^{r}, it must replace $a_{1,1}^{r+1}$ by some letter $a_{1,k}^{i}$ $(i \leqslant r+1)$, The group $\{T_{i_2}, T_{i_3}, \cdots, T_{i_{r-i+1}}\}$ leaves Γ_1^{i-1} invariant, $a_{1,1}^{i-1}$ and $a_{2,1}^{i-1}$ fixed, and has a constituent which is transitive in the letters $a_{1,k}^{x}$ $(k = 1, 2, \cdots, q_i;$ $x = i, i+1, \cdots, r)$, so that from it we may take a substitution S''' which replaces the letter $a_{1,k}^{i}$ that follows $a_{1,1}^{r+1}$ in $T_{i_1}T$ by a letter $a_{1,i_1'}^{r}$. Now $T_{i_1}TS'''$ replaces $a_{1,1}^{r+1}$ by $a_{1,i_1'}^{r}$; let α' be the system into which S''' changes α''. Since S''' is a substitution of Γ_1^{r}, α' satisfies the condition imposed on α''. Let S'' be a substitution of Γ_1^{i-1} that replaces the system α' by another (α) which includes the letter $a_{1,1}^{i-1}$. Finally let S' be a substitution of Γ_1^{i-1} the inverse of which replaces $a_{1,1}^{i-1}$ by the same letter as does $T_{i_1}TS'''S''$; S' leaves fixed the system α. Then $U = T_{i_1}TS'''S''S'$ leaves Γ_1^{i-1} invariant, $a_{1,1}^{i-1}$ fixed, replaces $a_{1,1}^{r+1}$ by $a_{1,i_1'}^{r}$, and changes one of the systems which have $a_{2,1}^{i-1}$ in common directly into α. Now $UT_{i_1'}^{-1} = (a_{1,1}^{i-1})(a_{1,1}^{r+1}) \cdots = S$, a substitution of Γ_2^{r}, which leaves the element $a_{2,1}^{i-1}$ fixed since $a_{1,1}^{i-1}$, one of the letters of $a_{2,1}^{i-1}$, is fixed. From $U = ST_{i_1'}$ it is clear that α must be one of the set of systems obtained from those of Γ_2^{i-1} by means of T_{i_1}.

To prove that any two of the $q_i \sum_{x=i}^{r+1} 1/q_x$ systems with one letter in common have q_{i+1} and only q_{i+1} in common, we have only to note that the series of transitive subgroups Γ_1^{x} $(x = i, i+1, \cdots, r)$ may be so chosen that any one of the systems involving the letter $a_{1,1}^{i-1}$ may be made to take the place of $a_{r-i+2,1}^{i-1}$. In fact $\{T_{i_1}, T_{i_2}, \cdots, T_{i_{r-i+1}}\}$ contains a substitution T which replaces any one

of the $q_i \sum_{x=i}^{r+1} 1/q_x$ systems which include $a_{1,1}^{i-1}$ by any other and permutes them only among themselves. Such a series of subgroups is Γ_1^1, Γ_1^2, $\cdots$, Γ_1^{i-1}, $T^{-1}\Gamma_1^i T$, $\cdots$, $T^{-1}\Gamma_1^r T$, G, and in it an arbitrary system plays the part of $a_{r-i+2,1}^{i-1}$.

It remains to be shown that the q_i systems which have $a_{1,1}^{i-1}$ but not $a_{2,1}^{i-1}$ in common contain no letter of the system $a_{2,1}^{i-1}$ except $a_{1,1}^{i-1}$. These q_i systems are those by which the q_i/q_r systems in the letters of Γ_2^{i-1}, which do not have $a_{3,1}^{i-1}$ in common, are replaced by $T_{i_1}(i_1 = 1, 2, \cdots, q_r)$. By hypothesis these q_i/q_r systems contain no letter of the system $a_{3,1}^{i-1}$, except $a_{2,1}^{i-1}$, and T_{i_1} permutes the letters $a_{1,1}^{i-1}$, $\cdots$, of $a_{3,1}^{i-1}$, among themselves, giving in place of the system $a_{2,1}^{i-1}$, q_r systems which have no letter in common with it except $a_{1,1}^{i-1}$. Hence none of these $(q_i/q_r)q_r = q_i$ systems have, besides $a_{1,1}^{i-1}$ any letter in common with $a_{2,1}^{i-1}$.

This theorem has been proved for the systems of q_r letters in Γ_1^{r-1}, and in consequence holds for the systems of q_{r-1} letters in Γ_1^{r-2}, and so on. In regard to H^1 we may now say that *systems of imprimitivity of q_i letters may be chosen in $q_i \sum_{x=i}^{r+1} 1/q_x$ distinct ways, for $i = 2, 3, \cdots, r-1, r$.*

If G is contained in a larger primitive group G' of degree n, G' is $n - \sum_{x=1}^{r} q_x + 1$ times transitive. Now

$$\sum_1^r q_x \leqq q_1\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \frac{1}{2^{r-1}}\right) \leqq 2q_1\left(1 - \frac{1}{2^r}\right) \leqq 2q_1\left(1 - \frac{1}{q_1}\right).$$

Hence the theorem:

A primitive group of degree n which contains a transitive subgroup of degree q_1 is at least $n - 2q_1 + 3$ times transitive.

We may give another incomplete but useful statement of our theorem:

If a primitive group G of degree n contains a transitive subgroup H of degree q_1, G is at least $n - q_1 - q_r(q_2 - 1)/(q_r - 1) + 1$ times transitive, q_2 being the greatest divisor of q_1 such that H has at least $q_2 + 1$ distinct arrangements of its letters in systems of imprimitivity of q_2 each, and q_r being the least divisor of q_1 such that H has at least $q_r + 1$ systems of q_r letters each with one letter in common, and not more than the one letter common to any two of them.

If both these conditions are not satisfied by the given group H, G is at least $n - q_1 + 1$ times transitive.

From $q_{i-1} \geqq q_i^2/q_{i+1}$, it follows that $q_r^{r-i+1} \leqq q_i \leqq q_2^{(r-i+1)(r-1)}$. Then

$$q_3 + q_4 + \cdots + q_r \leqq q_2^{\frac{r-2}{r-1}} + q_2^{\frac{r-3}{r-1}} + \cdots + q_2^{\frac{1}{r-1}} \leqq \frac{q_2 - 1}{q_2^{1/(r-1)} - 1} - 1 \leqq \frac{q_2 - 1}{q_r - 1} - 1.$$

Hence G, which is $n - \sum_{x=1}^{r+1} q_x$ times transitive, is at least

$$n - q_1 - q_2 - \frac{q_2 - 1}{q_r - 1} + 2 = n - q_1 - q_r \frac{q_2 - 1}{q_r - 1} + 1$$

times transitive.

This limit is attained by the holomorph of the Abelian group of degree 2^a and type $(1, 1, \cdots)$, which is triply transitive and has a transitive subgroup of degree 2^{a-1}, for which $q_2 = 2^{a-2}$ and $q_r = 2$.

Other useful conditions which Γ_1^{i-1} must satisfy may be obtained by considering the meaning of the multiple imprimitivity it exhibits.

Let J be an imprimitive group of degree n and order nm; let $a_1, a_2, \cdots, a_q$ be the letters of a certain system of imprimitivity of J. The substitutions, qm in number, that replace a_1 by $a_1, a_2, \cdots, a_q$ respectively, leave the system $a_1, \cdots, a_q$ fixed. The product of any two of them also has this property. No other substitution of J permutes the letters $a_1, \cdots, a_q$ among themselves. Then these qm substitutions form a group (H). Let there be another system of q letters in J which has the letters $a_1, a_2, \cdots, a_\alpha$ in common with the first, and the remaining letters $a'_{\alpha+1}, a'_{\alpha+2}, \cdots, a'_q$ distinct. The group H' which corresponds to this system has exactly $q\alpha$ substitutions in common with H: those which replace a_1 by $a_1, a_2, \cdots, a_\alpha$. These αm common substitutions form by themselves a group (F).

Now if a substitution T transforms J into itself, permutes the letters $a_2, a_3, \cdots, a_\alpha$ among themselves, leaves a_1 fixed, and replaces $a_{\alpha+1}, \cdots, a_q$ by $a'_{\alpha+1}, \cdots, a'_q$, it transforms F into itself and H into H'. We conclude that Γ_1^{i-1} *has a set of* $q_i \sum_{x=i}^{r+1} 1/q_x$ *subgroups of order* $q_i m_{i-1}$ *which have in common the subgroup* (*of order* m_{i-1}) *of* Γ_1^{i-1} *that leaves one letter fixed. Any two of these subgroups have in common also a subgroup of order* $q_{i+1} m_{i-1}$. This set of $q_i \sum_{x=i}^{r+1} 1/q_x$ subgroups of order $q_i m_{i-1}$ must be such that some isomorphism of Γ_1^{i-1} to itself brings any two we please into a 1, 1 correspondence. In particular, no one of them can be a characteristic subgroup of Γ_1^{i-1}. If one is invariant in Γ_1^{i-1}, all are invariant.

We recall that Γ_1^{r-1} admits $q_r + 1$ arrangements of its letters in systems of imprimitivity of q_r each such that $q_r + 1$ systems may be taken with one common letter and no two of which have more than this one letter in common. If $a_{1,1}^{r-1}$ be taken as the leading letter, the $q_r^2 - 1$ letters thus associated with it are all found in the larger system $a_{3,1}^{r-1}$. It may happen, as when $q_{r-1} = q_r^2$, that the q_r^2 letters in question form a system (A) of imprimitivity of Γ_1^{r-1}. The subgroup Δ' which leaves this system A fixed has a transitive constituent on the q_r^2 letters of A. We call this constituent group Δ and suppose all the other letters of Δ' erased. Now Δ has the properties of Γ_1^{r-1} in so far as systems of imprimitivity are concerned — even more extensive properties perhaps. Consider a substitution S of Δ which leaves fixed two letters $a_{1,1}^{r-1}$ and $a_{1,k}^{r-1}$. Let $S = (a_{1,\rho}^{r-1} a_{1,\sigma}^{r-1} \cdots) \cdots$. Since S can only permute among themselves the letters

of each of the systems to which $a_{1,1}^{r-1}$ belongs—and similarly for $a_{1,k}^{k-1}$—the letters $a_{1,\rho}^{r-1}$ and $a_{1,\sigma}^{r-1}$ belong to that system which contains both $a_{1,1}^{r-1}$ and $a_{1,k}^{r-1}$. Then S displaces at most most $q_r - 2$ letters, from which it readily follows that S is the identity. We conclude that when Δ is not regular, it is of class $q_r^2 - 1$ and has a characteristic transitive subgroup (Θ) which is regular.* This regular subgroup of order q_r^2 occurs in both cases, and Δ, when regular, may be called Θ for the sake of uniformity. This group Θ admits a $(q_r + 1)$ fold division into systems of imprimitivity, no two systems having more than one letter in common, hence its substitutions are distributed among $q_r + 1$ subgroups of order q_r, no two of which have an operator in common other than the identity.

Let $s_1 = 1, s_2, \cdots, s_{q_r}$ be the substitutions of one of these subgroups, one of whose substitutions is conjugate to some substitution t_j not in it. Let $t_1 = 1, t_2, \cdots, t_j, \cdots, t_{q_r}$ be that one of the $q_r + 1$ subgroups in question which includes t_j. Now every substitution of the group Θ is given by $s_\alpha t_\beta (\alpha, \beta = 1, 2, \cdots, q_r)$. But clearly $t_\beta^{-1} s_\alpha^{-1} s_i s_\alpha t_\beta \neq t_j$. Then each of the $q_r + 1$ subgroups is invariant, and since no two have anything in common but the identity, every substitution of one of these subgroups is commutative with all the substitutions not in it. Then the group Θ is Abelian. Let $q_r = mp^\alpha$, where, if possible, m is prime to p (p being a prime number). The Sylow subgroup of Θ of order $p^{2\alpha}$ must have $mp^\alpha + 1$ subgroups of order p^α with nothing in common but the identity. Then

$$1 + (mp^\alpha + 1)(p^\alpha - 1) = p^{2\alpha},$$

so that $m = 1$. If now we take the two subgroups of order $q_r = p^\alpha$ in which the subgroups composed of the operators of order p are of the lowest possible orders p^{k_1} and p^{k_2}, $k_1 \geqq k_2$, we get the inequality $p^{k_1 + k_2 - \alpha} \geqq p^{k_1}$, whence $k_1 = k_2 = \alpha$. Then Θ is of type $(1, 1, \cdots)$.† The elementary group of order 16 has 5 subgroups of order 4, no two of which have anything but the identity in common. It is found in the quintuply transitive group of degree 24 of MATHIEU.‡

Another assumption we may make in regard to Γ_1^{r-1} is that there is in it an imprimitive system of $q_r^2 + q_r$ letters including $a_{1,1}^{r-1}$ and the $q_r^2 - 1$ letters associated with $a_{1,1}^{r-1}$ in imprimitive systems of q_r letters each. Just as in the preceding case there is a subgroup Δ' that has a transitive constituent Δ of degree $q_r^2 + q_r$, which we now investigate.

Let S be a substitution of Δ that leaves fixed two letters $a_{1,1}^{r-1}$ and $a_{1,k}^{r-1}$. By

* FROBENIUS, Berliner Sitzungsberichte, 1901, pp. 1216–1230, and 1902, pp. 455–459.

† Cf. MILLER, Bulletin of the American Mathematical Society, vol. 12 (1906), pp. 446–449. This is the proof Professor MILLER mentions in the note at the bottom of page 449.

‡ MATHIEU, Journal de Mathématiques, ser. 2, vol. 18 (1873), pp. 25–46.

MILLER, Bulletin de la Société mathématique de France, vol. 28 (1900), pp. 266–267.

the method used above we know that the only letters which S can displace are the $q_r - 2$ other letters of a system which includes both $a_{1,1}^{r-1}$ and $a_{1,k}^{r-1}$, the q_r letters not in a system with $a_{1,1}^{r-1}$ and the q_r not in a system with $a_{1,k}^{r-1}$, $3q_r - 2$ at most. But if $S = (a_{1,\rho}^{r-1} a_{1,\sigma}^{r-1} \cdots) \cdots$, it must replace every system of q_r letters in which $a_{1,\rho}^{r-1}$ is found by a system in which $a_{1,\sigma}^{r-1}$ is present. Then S displaces at least $q_r^2 - q_r + 2$ letters. Now $q_r^2 - q_r + 2 > 3q_r - 2$ when $q_r > 2$. It is obvious that when $q_r = 2$, Δ is regular. Then for all values of q_r, Δ is either regular or of class $q_r^2 + q_r - 1$. In the latter case Δ contains a regular characteristic subgroup. Then the regular subgroup Θ of order $q_r^2 + q_r$ is always found in the constituent Δ of Δ'. Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_{q_r+1}$ be the subgroups with nothing in common but the identity which correspond to the $q_r + 1$ systems of imprimitivity of q_r letters each in question. In Θ systems of q_r letters each are permuted according to a transitive group of degree $q_r + 1$. All the substitutions which lie in the subgroups $\Lambda_1, \cdots$ leave fixed at least one of these systems of imprimitivity. This is seen by applying to Θ all the substitutions of its conjoin.* Then Θ has just q_r substitutions which displace all $q_r + 1$ systems. The group of degree $q_r + 1$ in the systems is of class q_r. For it is a characteristic property of groups of "class $n - 1$" that they have just $n - 1$ substitutions of degree n.† Hence Θ has a characteristic subgroup Π which leaves fixed none of the $(q_r + 1)^2$ possible systems. It is clear that $\Lambda_1, \cdots$ are conjugate under Π. Consequently the group in the systems is of order $(q_r + 1)q_r$ and doubly transitive. Hence Π is Abelian of type $(1, 1, \cdots)$.‡ If $q_{r-1} = q_r(q_r + 1)$, we have $q_{r-1} + q_r + 1 = p^{2a}$. The doubly transitive group of order 432, degree 9 and class 6 is a case in point. The transitive subgroup of degree 6 in it is non-cyclic.

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* JORDAN, *Traité des Substitutions*, 1870, p. 80.

† BURNSIDE, Proceedings of the London Mathematical Society, vol. 32 (1900), pp. 240-246.

‡ JORDAN, Journal de Mathématiques, ser. 2, vol. 17 (1872), pp. 351-367; FROBENIUS, l. c.

ZUR THEORIE DER VOLLSTÄNDIG REDUCIBLEN GRUPPEN, DIE ZU EINER GRUPPE LINEARER HOMOGENER SUBSTITUTIONEN GEHÖREN*

(AUS EINEM BRIEFE AN HERRN ALFRED LOEWY)

VON

LUDWIG STICKELBERGER

Die Grundlage des Beweises für Ihren kürzlich veröffentlichten Satz über die grössten vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören, bildet, wie Sie wissen, der in § 2 Ihrer Arbeit (*Über die vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören*, Transactions, vol. 6, pp. 504–533), bewiesene Hilfssatz. Zu seinem Beweise berufen Sie sich auf einen Aufsatz von Herrn MASCHKE. Bei der Lectüre Ihrer Arbeit fand ich, dass man an die Stelle Ihres Hilfssatzes folgenden Satz treten lassen kann, den ich verhältnismässig einfach ohne Heranziehung fremder Untersuchungen bewies. Es seien

$$(\mathfrak{A}) \quad \begin{matrix} \mathfrak{A}_{11} & 0 \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{matrix} \qquad \text{und} \qquad (\mathfrak{B}) \quad \begin{matrix} \mathfrak{b}_{11} & 0 \\ \mathfrak{b}_{21} & \mathfrak{b}_{22} \end{matrix}$$

zwei ähnliche Gruppen linearer homogener Substitutionen in n Variablen; ihre Bestandteile $\mathfrak{A}_{11}$ und $\mathfrak{b}_{11}$ mögen Gruppen in ν bezüglich π Variablen sein, $\nu < n$, $\pi < n$. Die Substitutionen von $\mathfrak{A}$ mögen lauten:

$$(1) \qquad y_i = \sum_{s=1}^{s=n} a_{is}^{(t)} y_s' \qquad (i=1, 2, \cdots, n);$$

t habe hierbei die Werte $1, 2, 3, \cdots$, und charakterisire die einzelnen Substitutionen der Gruppe $\mathfrak{A}$. Da die Gruppe $\mathfrak{A}$ die Form $(\mathfrak{A})$ hat, werden sich in (1) die ersten ν Variablen in der Form:

$$(2) \qquad y_i = \sum_{k=1}^{k=\nu} a_{ik}^{(t)} y_k' \qquad (i=1, 2, \cdots, \nu)$$

transformiren, es wird also für $i \leqq \nu$, $k > \nu$ stets $a_{ik}^{(t)} = 0$ sein.

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Die Substitutionen von $\mathfrak{B}$ mögen

$$z_i = \sum_{s=1}^{s=n} b_{is}^{(t)} z'_s \qquad (i=1, 2, \cdots, n) \tag{3}$$

lauten; der obere Index t, der die Werte $1, 2, 3, \cdots$ annehmen möge, soll dabei auf die Zuordnung der Substitutionen von $\mathfrak{A}$ und $\mathfrak{B}$ hinweisen. Die ersten π Variablen, welche die Substitutionen der Gruppe $\mathfrak{b}_{11}$ bestimmen, werden Substitutionen der Form:

$$z_i = \sum_{k=1}^{k=\pi} b_{ik}^{(t)} z'_k \qquad (i=1, 2, \cdots, \pi) \tag{4}$$

erleiden.

Da die Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ ähnlich sind, so wird eine Transformation:

$$z_i = \sum_{s=1}^{s=n} q_{is} y_s \qquad (i=1, 2, \cdots, n) \tag{5}$$

von nicht verschwindender Determinante $|q_{ik}|$ existiren, die mit ihrer cogredienten

$$z'_i = \sum_{s=1}^{s=n} q_{is} y'_s \qquad (i=1, 2, \cdots, n) \tag{5'}$$

alle Substitutionen der Gruppe $\mathfrak{A}$ in die entsprechenden von $\mathfrak{B}$ überführt.

Ich habe mich der gleichen Bezeichnungen wie Sie auf Seite 514 bedient; nur habe ich statt Ihres Buchstabens g_1 den griechischen Buchstaben π verwandt, ferner setze ich *nicht* voraus, dass $\mathfrak{b}_{11}$ eine irreducible Gruppe ist.

Anstelle Ihres Hilfssatzes beweise ich folgenden Satz:

Sind die vorhin charakterisirten Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ ähnliche Gruppen linearer homogener Substitutionen und bestehen zwischen $y_1, y_2, \cdots, y_\nu$ und $z_1, z_2, \cdots, z_\pi$ genau ρ linear unabhängige lineare homogene Relationen, so haben die Gruppen $\mathfrak{A}_{11}$ und $\mathfrak{b}_{11}$ eine Gruppe von ρ Variablen gemeinsam, d. h. $\mathfrak{A}_{11}$ und $\mathfrak{b}_{11}$ sind zwei Gruppen der Form:

$$\begin{matrix} \bar{\mathfrak{A}}_{11} & 0 \\ \bar{\mathfrak{A}}_{21} & \bar{\mathfrak{A}}_{22} \end{matrix} \quad \textit{und} \quad \begin{matrix} \bar{\mathfrak{A}}_{11} & 0 \\ \mathfrak{b}'_{21} & \mathfrak{b}'_{22} \end{matrix}$$

ähnlich; hierbei definirt $\bar{\mathfrak{A}}_{11}$ eine Gruppe linearer homogener Substitutionen in ρ Variablen.

Die ρ linearen homogenen unabhängigen Relationen zwischen $y_1, y_2, \cdots, y_\nu$ und $z_1, z_2, \cdots, z_\pi$, die nach (5) lineare homogene Functionen von $y_1, y_2, \cdots, y_n$ sind, mögen lauten:

$$e_1^{(\lambda)} y_1 + e_2^{(\lambda)} y_2 + \cdots + e_\nu^{(\lambda)} y_\nu + f_1^{(\lambda)} z_1 + f_2^{(\lambda)} z_2 + \cdots + f_\pi^{(\lambda)} z_\pi = 0 \quad (\lambda=1, 2, \cdots, \rho). \tag{6}$$

Setzt man:

$$Y_\lambda = e_1^{(\lambda)} y_1 + e_2^{(\lambda)} y_2 + \cdots + e_\nu^{(\lambda)} y_\nu \qquad (\lambda = 1, 2, \cdots, \rho), \tag{7}$$

$$Z_\lambda = -f_1^{(\lambda)} z_1 - f_2^{(\lambda)} z \cdots - f_\pi^{(\lambda)} z_\pi \qquad (\lambda = 1, 2, \cdots, \rho), \tag{8}$$

so gehen die ρ Relationen (6) über in:

$$Y_1 = Z_1,\ Y_2 = Z_2,\ \cdots,\ Y_\rho = Z_\rho,$$

wo die linken Seiten von den y allein, die rechten Seiten von den z allein abhängen. Sowohl die ρ Functionen $Y_1, Y_2, \cdots, Y_\rho$ als auch die ρ Functionen $Z_1, Z_2, \cdots, Z_\rho$ sind wegen der Bedeutung von ρ linear unabhängig. Zu den ρ Functionen $Y_1, Y_2, \cdots, Y_\rho$ der Variablen $y_1, y_2, \cdots, y_\nu$ mögen noch $\nu - \rho$ weitere lineare homogene Functionen $Y_{\rho+1}, Y_{\rho+2}, \cdots, Y_\nu$ der $y_1, y_2, \cdots, y_\nu$, und zu den ρ Functionen $Z_1, Z_2, \cdots, Z_\rho$ der $z_1, z_2, \cdots, z_\pi$ noch $\pi - \rho$ weitere lineare homogene Functionen $Z_{\rho+1}, Z_{\rho+2}, \cdots, Z_\pi$ der $z_1, z_2, \cdots, z_\pi$ so zugefügt werden, dass $Y_1, Y_2, \cdots, Y_\nu$ und ebenso $Z_1, Z_2, \cdots, Z_\nu$ unabhängig sind.

Entsprechend den ρ Relationen (6) bestehen für die cogredienten Variablen y' und z' genau die gleichen Relationen, in denen für y und z die Veränderlichen y' und z' stehen. Führt man in entsprechender Weise wie

$$Y_\lambda \quad (\lambda = 1, 2, \cdots, \nu) \qquad \text{und} \qquad Z_\lambda \quad (\lambda = 1, 2, \cdots, \pi)$$

die cogredienten Veränderlichen

$$Y'_\lambda \quad (\lambda = 1, 2, \cdots, \nu) \qquad \text{und} \qquad Z'_\lambda \quad (\lambda = 1, 2, \cdots, \pi)$$

ein, die wie Y_λ und Z_λ aus den y und z in genau gleicher Weise aus den y' und z' gebildet sind, so sind einerseits $Y'_1, Y'_2, \cdots, Y'_\nu$ und andererseits $Z'_1, Z'_2, \cdots, Z'_\pi$ linear unabhängig, und zwischen ihnen bestehen nur die ρ Relationen:

$$Y'_1 = Z'_1,\ Y'_2 = Z'_2,\ \cdots,\ Y'_\rho = Z'_\rho.$$

Durch Einführung der neuen Veränderlichen geht die Gruppe $\mathfrak{A}_{11}$ in eine ähnliche Gruppe A_{11} in den Veränderlichen Y und Y' über; ihre Substitutionen sind etwa von der Form:

$$Y_i = \sum_{k=1}^{k=\nu} \bar{a}_{ik}^{(t)} Y'_k \qquad (i = 1, 2, \cdots, \nu). \tag{9}$$

Die Gruppe $\mathfrak{b}_{11}$ wird einer Gruppe $\mathfrak{B}_{11}$ in den Variablen Z und Z' ähnlich, deren Substitutionen etwa lauten:

$$Z_i = \sum_{k=1}^{k=\pi} \bar{b}_{ik}^{(t)} Z'_k \qquad (i = 1, 2, \cdots, \pi). \tag{10}$$

Für jede Substitution der Gruppe A_{11} ist nach (9) Y_1 eine lineare homogene

Function von $Y'_1, Y'_2, \cdots, Y'_\nu$. Da für jede Substitution der Gruppe β_{11} die Variable Z_1 nach (10) eine lineare homogene Function von $Z'_1, Z'_2, \cdots, Z'_\pi$ ist, folgt aus der Relation $Y_1 = Z_1$, dass Y_1 auch eine lineare homogene Function von $Z'_1, Z'_2, \cdots, Z'_\pi$ ist. Hieraus ergiebt sich, dass für jede Substitution der Gruppe A_{11} die Variable Y_1 ausschliesslich lineare homogene Function von $Y'_1, Y'_2, \cdots, Y'_\rho$ ist. Sie werden dies unmittelbar einsehen. Es gilt nämlich allgemein, dass jede lineare homogene Function von $Y'_1, Y'_2, \cdots, Y'_\nu$, die gleichzeitig lineare homogene Function von $Z'_1, Z'_2, \cdots, Z'_\pi$ ist, eine lineare homogene Function von $Y'_1, Y'_2, \cdots, Y'_\rho$ allein ist. Dies folgt einfach daraus, dass jede lineare homogene Function der eben beschriebenen Art eine Relation zwischen $Y'_1, Y'_2, \cdots, Y'_\nu$ und $Z'_1, Z'_2, \cdots, Z'_\pi$ zur Folge hat. Diese muss aber, da zwischen $Y'_1, Y'_2, \cdots, Y'_\nu$ und $Z'_1, Z'_2, \cdots, Z_\pi$ nur die ρ Relationen

$$Y'_1 = Z'_1,\ Y'_2 = Z'_2, \cdots,\ Y'_\rho = Z'_\rho$$

bestehen, eine Folge dieser ρ Beziehungen sein, also von $Y'_{\rho+1}, Y'_{\rho+2}, Y'_\nu$ und $Z'_{\rho+1}, Z'_{\rho+2}, \cdots, Z'_\pi$ frei sein.

Genau das gleiche Resultat wie für Y_1 gilt offenbar für $Y_2, Y_3, \cdots, Y_\rho$, daher lauten die ersten ρ Transformationen der Gruppe A_{11}:

$$Y_i = \sum_{k=1}^{k=\rho} \bar{a}^{(l)}_{ik} Y'_k \qquad (i = 1, 2, \cdots, \rho).$$

Aus

$$Y_\lambda = Z_\lambda,\ Y'_\lambda = Z'_\lambda \qquad (\lambda = 1, 2, \cdots, \rho)$$

ergiebt sich, dass die ersten ρ Transformationen von β_{11}:

$$Z_i = \sum_{k=1}^{k=\rho} \bar{a}^{(l)}_{ik} Z'_k \qquad (i = 1, 2, \cdots, \rho)$$

lauten. Hiermit ist mein Satz bewiesen.

Ist $\mathfrak{b}_{11}$ im besonderen eine irreducible Gruppe, so muss wegen der Irreducibilität von $\mathfrak{b}_{11}$ für $\rho > 0$ die Gruppe $\overline{\mathfrak{A}}_{11}$ zu $\mathfrak{b}_{11}$ ähnlich sein. In dem Falle, dass $\mathfrak{b}_{11}$ eine irreducible Gruppe ist, bezeichne ich die Anzahl ihrer Variablen statt mit dem Buchstaben π mit dem nämlichen Buchstaben g_1 wie Sie. Da dann $\mathfrak{b}_{11}$ ein Bestandteil von $\mathfrak{A}_{11}$ ist, müssen die Variablen $z_1, z_2, \cdots, z_{g_1}$ von $\mathfrak{b}_{11}$ lineare homogene Functionen der Variablen $y_1, y_2, \cdots, y_\nu$ von $\mathfrak{A}_{11}$ sein, d. h. es werden Relationen:

$$z_i = \sum_{s=1}^{s=\nu} r_{is} y_s \qquad (i = 1, 2, \cdots, g_1)$$

bestehen; hierbei sind die $\nu \cdot g_1$ Grössen r_{is} Constante. Aus (5) ergiebt sich dann:

$$\sum_{s=1}^{s=n} q_{is} y_s = \sum_{s=1}^{s=\nu} r_{is} y_s \qquad (i = 1, 2, \cdots, g_1). \tag{11}$$

Da $y_1, y_2, \cdots, y_n$ linear unabhängige Variablen sind, müssen in den g_1 Relationen (11) die Grössen $r_{ik} = q_{ik}$ für $i \leqq g_1$, $k = 1, 2, \cdots, \nu$ und $q_{ik} = 0$ für $i \leqq g_1$, $k > \nu$ werden. Ist $\mathfrak{b}_{11}$ also eine irreducible Gruppe, so lauten für $\rho > 0$ die ersten g_1 Gleichungen der überführenden Transformation (5):

$$z_i = \sum_{k=1}^{k=\nu} q_{ik} y_k \qquad (i = 1, 2, \cdots, g_1).$$

Dies ist der von Ihnen auf Seite 515 ausgesprochene Satz.

Das angewandte Beweisverfahren führt keine Irrationalitäten ein; daher gilt es auch für den Fall, dass die Coefficienten sämtlicher Substitutionen der Gruppen $\mathfrak{A}$ und $\mathfrak{B}$ einem Zahlkörper Ω angehören und man sich, wie Sie es im § 6 Ihrer Arbeit tun, ausschliesslich im Körper Ω bewegt.

ON COMMUTATIVE LINEAR ALGEBRAS IN WHICH DIVISION IS ALWAYS UNIQUELY POSSIBLE*

BY

L. E. DICKSON

1. We consider commutative linear algebras in $2n$ units, with coördinates in a general field F, such that n of the units define a sub-algebra forming a field $F(J)$. The elements of the algebra may be exhibited compactly in the form $A + BI$, where A and B range over $F(J)$. As multiplication is not associative in general, A and B do not play the rôle of coördinates, so that the algebra is not binary in the usual significance of the term.† Nevertheless, by the use of this binary notation, we may exhibit in a very luminous form the multiplication-tables of certain algebras in four and six units, given in an earlier paper.‡ Proof of the existence of the algebras and of the uniqueness of division now presents no difficulty. The form of the corresponding algebra in $2n$ units becomes obvious. After thus perfecting and extending known results, we attack the problem of the determination of all algebras with the prescribed properties. An extensive new class of algebras is obtained.

2. Consider first algebra (VIII) of the paper cited. Let $x^2 - dx + c$ be irreducible in F, c being a not-square. The units are $1, I, J, K$, with

$$I^2 = J, \qquad IJ = JI = K, \qquad IK = KI = c,$$

$$J^2 = dJ - c, \qquad JK = KJ = dK - cI, \qquad K^2 = cd - cJ.$$

The elements $Y = y + \eta J$, y and η ranging over F, form a field $F(J)$. In this field $x^2 - dx + c = 0$ has the roots J and $J' = c/J$. Set $Y' = y + \eta J'$. Then if B and Y are any two elements of $F(J)$, we readily find that

$$(1) \qquad B(YI) = (BY)I, \qquad (BI)(YI) = B'Y'J.$$

* Presented to the Society at the New Haven summer meeting, September 3, 1906. Received for publication July 2, 1906.

† A binary algebra in which division is always possible is a field.

‡ Transactions, vol. 7 (1906), pp. 370-390, algebras (VIII), (X).

The rule of multiplication of any two elements of the algebra is thus

$$(2)\quad (A+BI)(X+YI)=R+SI,\quad R\equiv AX+B'Y'J,\quad S\equiv BX+AY.$$

To show that division is always uniquely possible, we let A, B, R, S be any marks of $F(J)$ such that A and B are not both zero, and prove that marks X and Y can be uniquely determined so as to satisfy (2). Eliminating X, we get

$$(3)\qquad BB'Y'J-A^2Y=C,\qquad C\equiv BR-AS.$$

It suffices to show that Y can be found in $F(J)$ to satisfy (3) and

$$(3')\qquad -A'^2Y'+BB'YJ'=C'.$$

This can be done since $cB^2B'^2-A^2A'^2\neq 0$, c being a not-square in F.

3. We next indicate the essential point in the proof that algebra (X) on six units, treated in § 5 of the paper cited, can be exhibited in the luminous form (2). Now j (our present J) is a root of

$$(4)\quad x^3-(2d_3+d_3d_5-c^{-1}d_1^2)x^2-(d_1-d_1d_5-d_3^2-d_3^2d_5+c^{-1}d_1^2d_3)x-c=0,$$

where

$$c^2+c^2d_5+c^2d_5^2+cd_1d_3+cd_1d_3d_5-d_1^3=0.$$

Equation (4) *is a normal cubic.* Indeed, it has a second root l/j, viz.,

$$(5)\qquad j'=d_5^{-1}(j-d_3-d_1j^{-1}).$$

It is now a simple matter to verify (1), with $B=b_1+b_2j+b_3l$, etc.

4. The generalization to $2n$ units is now obvious. Consider any uniserial abelian equation, with coefficients in F,

$$(6)\qquad x^n-c_1x^{n-1}+c_2x^{n-2}-\cdots\pm c_n=0\quad (c_n \text{ a not-square in } F),$$

viz., an equation irreducible in F and having the roots

$$(7)\qquad J,\ J'=\theta(J),\ J''=\theta^2(J),\ \cdots,\ J^{(n-1)}=\theta^{n-1}(J)\quad [\theta^n(J)\equiv J],$$

where θ is a polynomial with coefficients in F. If A is a polynomial in J with coefficients in F, we denote $A(J')$ by A', $A(J'')$ by A'', etc. Then J is a not-square in the field $F(J)$. For, if $J=f^2$, then $J'=f'^2$, etc., so that $c_n\equiv JJ'J''\cdots$ would be the square of $ff'f''\cdots$, which is a mark of F. Consider the algebra with the $2n$ units

$$(8)\qquad J^r,\qquad IJ^r\qquad (r=0,1,\cdots,n-1),$$

subject to the multiplication-table implied by (2). To show that division is always uniquely possible, it suffices to prove that there exists a solution Y in $F(J)$ of equation (3). Writing down the $n-1$ equations obtained from (3) by passing to the conjugates, and solving the systems of n linear equations, we get

$$\Delta Y = \sum_{s=0}^{n-1} C^{(s)} BB'J \cdot B'B''J' \cdots B^{(s-1)} B^{(s)} J^{(s-1)} [A^{(s+1)}]^2 \cdots [A^{(n-1)}]^2,$$

$$\Delta = c_n B^2 B'^2 \cdots [B^{(n-1)}]^2 - A^2 A'^2 \cdots [A^{(n-1)}]^2.$$

Since c_n is a not-square in F, while A and B are not both zero, we have $\Delta \neq 0$. The resulting value of Y is seen to satisfy (3).

In conclusion, we note that if F is the $GF[p^m]$, $p > 2$, the algebra exists, since there are irreducible equations (6) with c_n a not-square in F. Indeed, $c_n = J^t$, $t \equiv 1 + p^m + p^{2m} + \cdots + p^{m(n-1)}$, so that c_n is a not-square in F if (and only if) $J^{\frac{1}{2}(p^{nm}-1)} = -1$, i. e., if J is a not-square in the $GF[p^{nm}]$. It thus suffices to take as (6) a primitive irreducible equation in F.

5. We pass to the problem of the determination of all commutative algebras with $2n$ units (8), where J is a root of (6) and $I^2 = J$, while in place of (1) the law of multiplication is*

$$B(YI) = lBY + mB'Y' + I(uBY + wB'Y'), \tag{9}$$

$$(BI)(YI) = LBY + MB'Y' + I(UBY + WB'Y'), \tag{10}$$

$l, \cdots, W$ being fixed marks of $F(J)$, the same for every B, Y.

In (9) set $Y = 1$, $B = 1$; then $Y = 1$, $B = J$. Hence

$$l + m = 0, \quad u + w = 1; \qquad lJ + mJ' = 0, \quad uJ + wJ' = J.$$

But $J \neq J'$. Hence $l = m = w = 0$, $u = 1$. In (10) set $B = Y = 1$, and apply $I^2 = J$. Hence $L + M = J$, $U + W = 0$. Thus (9) and (10) become

$$B(YI) = (BY)I,$$

$$(BI)(YI) = LBY + (J - L)B'Y' + IU(BY - B'Y').$$

The rule of multiplication of any two elements of the algebra is thus

$$(A + BI)(X + YI) = R + SI,$$

$$R \equiv AX + LBY + (J - L)B'Y', \tag{11}$$

$$S \equiv BX + AY + U(BY - B'Y').$$

* The writer conjectures that every commutative linear algebra in four units in which division is unique satisfies these assumptions. By a tedious computation, this was established for the case in which F is the $GF[3]$.

Let A, B, R, S be any given marks of $F(J)$ such that A and B are not both zero. We seek the conditions under which marks X and Y of $F(J)$ can be uniquely determined so that (11) shall hold. By eliminating X, we are led to the equivalent problem to determine the conditions under which there is an unique solution Y in $F(J)$ of

$$YG + Y'H = C,$$

$$G \equiv A^2 + UAB - LB^2, \tag{12}$$

$$H \equiv (L - J)BB' - AB'U,$$

where $C \equiv AS - BR$ is a given mark. In (12) and

$$Y'G' + Y''H' = C', \qquad Y''G'' + Y'''H'' = C'', \cdots$$

$$Y^{(n-1)}G^{(n-1)} + YH^{(n-1)} = C^{(n-1)},$$

the determinant of the coefficients of $Y, Y', \cdots, Y^{(n-1)}$ equals

$$\Delta \equiv GG'G''\cdots G^{(n-1)} - (-1)^n HH'H''\cdots H^{(n-1)}. \tag{13}$$

If $\Delta \neq 0$, the values of $Y, \cdots, Y^{(n-1)}$, obtained by solving these n equations as linear equations, are seen to be conjugate. *Hence division is always uniquely possible if and only if* $\Delta = 0$ *implies* $A = B = 0$.

It will be convenient to set

$$E = L + \tfrac{1}{4}U^2, \qquad \alpha = A + \tfrac{1}{2}UB. \tag{14}$$

Then

$$G = \alpha^2 - EB^2, \qquad H = (E + \tfrac{1}{4}U^2 - J)BB' - U\alpha B'. \tag{15}$$

We first show that if $G = H = 0$ then $\alpha = B = 0$. For, if $B \neq 0$, then $\alpha = \kappa B$, $\kappa^2 = E$, κ in $F(J)$. Then $H = BB'\{\kappa - \tfrac{1}{2}U)^2 - J\}$. The last factor does not vanish since J is a not-square in $F(J)$.

Next, let $\Delta = 0$, $G \neq 0$. Then $H = \tau G$, where

$$\tau\tau'\tau''\cdots\tau^{(n-1)} = (-1)^n. \tag{16}$$

We may make an important normalization. Set

$$\alpha = \alpha_1 x, \qquad B = B_1 x, \qquad G_1 = \alpha_1^2 - EB_1^2,$$

$$H_1 = (E + \tfrac{1}{4}U^2 - J)B_1B_1' - U\alpha_1 B_1'.$$

Then $H = H_1 xx'$, $G = G_1 x^2$, so that $H_1 = G_1 \tau x/x'$. It can be shown that, in view of (16), x can be so chosen in $F(J)$ as to make $\tau x/x' = (-1)^n$.

After this normalization, we have $H=(-1)^nG$. For $D=\alpha+\frac{1}{2}(-1)^nUB'$, this becomes

$$(17)\quad D^2-[B+(-1)^nB'][EB+\tfrac{1}{4}(-1)^nU^2B']+(-1)^nJBB'=0.$$

Hence division is uniquely possible in the algebra if and only if $D=0$, $B=0$ *is the only set of solutions in the field* $F(J)$ *of equation* (17).

We have therefore reduced a problem on homogeneous forms of degree $2n$ in $2n$ variables to a problem on quadratic forms.

6. THEOREM. *For* $n=2$, *division is uniquely possible if*

$$(18)\quad E=\frac{1}{4c_2}J^2U'^2,$$

$$(19)\quad \epsilon^2-4c_2U^2U'^2=\begin{cases} a\ not\text{-}square^*\ in\ F(J) \\ or\ a\ square\ in\ F \end{cases}\qquad (\epsilon\equiv J'U^2+JU'^2-4c_2).$$

When F *is a finite field, division is uniquely possible in no further algebras* (11) *except the field-algebra defined by* $U=0$, $E=L=J$.

The first part of the theorem is readily proved. We may set $U\neq 0$, since the case $U=0$, $E=L=0$, has been treated in § 2. If we multiply the conjugate of (17) by J/J' and apply $JJ'=c_2$, we find that the final terms are the same as those in (17); hence $D^2=D'^2J/J'$. Thus $J'D^2$ is an element e of F. But $D^2=(d+\delta J)^2=Je/c_2$ gives $d^2-c_2\delta^2=0$, from which $d=\delta=0$. We next find the conditions under which (17), with $D=0$, has a root $B\neq 0$ in $F(J)$. Let $B=XB'$, so that $XX'=1$. Multiplying (17) by $-4c_2$, completing the square in X and applying (18), we get

$$(20)\quad (2XJU'^2+\epsilon)^2=\epsilon^2-4c_2U^2U'^2,$$

where ϵ, given by (19), is a mark of F. The second member must equal the square of a mark R of $F(J)$. Thus $X=(R-\epsilon)/2JU'^2$. Since $XX'=1$,

$$(R-\epsilon)(R'-\epsilon)=4c_2U^2U'^2\equiv(R-\epsilon)(-R-\epsilon).$$

Hence $R'=-R$, so that the second member of (20) is a not-square in F.

Inversely, if the second member of (20) is a square R^2 in $F(J)$, but a not-square in F, there exists a mark $X=x+yJ$ in $F(J)$ which satisfies (20) and $XX'=1$. Then $B=XB'$ is satisfied by a mark $B=b+\beta J\neq 0$ in $F(J)$. Indeed, the determinant of the coefficients of b and β in the equivalent two linear equations is $x^2+xyc_1+y^2c_2-1$, which is zero since $XX'=1$.

When (18) *holds,* $D=0$, $B=0$ *is the only set of solutions of* (17), *for* $n=2$, *if and only if condition* (19) *is satisfied.*

* The first alternative does not occur when F is a finite field.

To prove the second part of the theorem it now suffices to show that either $U=0$, $E=J$, or else that condition (18) holds. We show that in the remaining cases (17) has a set of solutions $D=d+\delta J$, $B=b+\beta J$, $b\neq 0$. If t is a mark of F, then tB, tD are solutions when B, D are; hence we set $b=1$. Set also

$$E=e+fJ, \qquad \tfrac{1}{4}U^2=r+sJ. \tag{21}$$

Hence (17) is equivalent to the two equations

$$d^2-\delta^2 c_2-(2+\beta c_1)(e+r+\beta r c_1-\beta f c_2+\beta s c_2)=0, \tag{22}$$

$$2d\delta+\delta^2 c_1-(2+\beta c_1)(f+s+\beta e+\beta f c_1-\beta r)+1+\beta c_1+\beta^2 c_2=0. \tag{23}$$

For brevity we treat in detail only the case in which F is the $GF[p^m]$, p^m of the form $4k+1$. Then if ν is a not-square, so is also $-\nu$. Then * we may set $c_1=0$, $c_2=-\nu$, so that $J^2=\nu$.

Let first $f=s$, so that β does not occur in (22). Set

$$\tau=\beta\nu+e-r, \qquad \gamma=2e+2r, \qquad \kappa=\nu-4\nu s+(e-r)^2. \tag{24}$$

Then (22) and (23) become

$$d^2+\nu\delta^2-\gamma=0, \qquad \tau^2-\kappa-2\nu d\delta=0. \tag{25}$$

If $d^2+\nu\delta^2=d_1^2+\nu\delta_1^2$ and $d\delta=d_1\delta_1$, we readily find that

$$(d^2,\ \delta^2)=(d_1^2,\ \delta_1^2) \qquad \text{or} \qquad (\nu\delta_1^2,\ \nu^{-1}d_1^2).$$

In the second case, $d=\delta=d_1=\delta_1=0$. Hence if $\gamma\neq 0$, then $d=\pm d_1$, $\delta=\pm\delta_1$. But there are exactly p^m+1 sets of solutions d, δ of $d^2+\nu\delta^2=\gamma$ $(\gamma\neq 0)$ in the $GF[p^m]$. Hence there are $\frac{1}{2}(p^m+1)$ distinct values of the product $d\delta$ of such solutions. But only $\frac{1}{2}(p^m-1)$ marks are not-squares. Hence there is at least one set of solutions d, δ of (25_1) for which $\kappa+2\nu d\delta$ is a square, so that (25_2) is solvable for τ. Hence if $f=s$ then $e=-r$ by (24), so that (18) holds.

Let next $f\neq s$ and set, for brevity,

$$\begin{aligned} &\sigma=s-f, \qquad g=e+r, \qquad t=-4f-2\sigma+1, \\ &h=2g-2\sigma(2r-g), \qquad l=-4\nu t-4(2r-g)^2. \end{aligned} \tag{26}$$

On eliminating β between equations (22) and (23) with $c_1=0$, we get

$$(d^2+\nu\delta^2-h)^2=\sigma^2(8\nu d\delta-l) \qquad (\sigma\neq 0). \tag{27}$$

* For remarks on the principle that any quaternary algebra of the kind considered is equivalent to one in which the irreducible quartic $x^4-c_1x^3+c_2=0$ is any chosen one, see paper cited in § 1.

We wish to prove that $h=0$, $l=4\nu\sigma^2$ is the only set of values of the parameters h, l for which (27) is not solvable for d, δ. It suffices to prove this for the case $\sigma^2=1$ in view of the normalization

$$d=d_1\sigma, \qquad \delta=\delta_1\sigma, \qquad h=h_1\sigma^2, \qquad l=l_1\sigma^2.$$

The normalized problem is thus to find the values of the parameters h, l for which there is no set of solutions ρ, d, δ in F of the simultaneous equations

$$d^2+\nu\delta^2-h=\rho, \qquad 8\nu d\delta-l=\rho^2. \tag{28}$$

Such a set of solutions exists if and only if

$$\phi\equiv\rho+h+\tfrac{1}{4}\nu^{-1}(\rho^2+l)J$$

is a square in $F(J)$, viz., the square of $d+\delta J$. Hence ϕ must be a not-square and therefore ϕJ a square in $F(J)$. Thus must

$$f_\rho\equiv\tfrac{1}{4}(\rho^2+l)+(\rho+h)J=\text{square in } F(J) \text{ for every } \rho \text{ in } F. \tag{29}$$

Hence $f_\rho f'_\rho$ must be a square in F, viz.,

$$(\rho^2+l)^2-16\nu(\rho+h)^2=\text{square in } F \text{ for every } \rho. \tag{30}$$

It can be shown that this condition requires that the left member be algebraically a perfect square in ρ, so that $h=0$, $l=4\nu$. Restoring σ, we have $h=0, l=4\nu\sigma^2$. Let first $\sigma=-1$. Then, by (26),

$$r=0, \qquad t=-1-\epsilon^2/\nu, \qquad f=1+\epsilon^2/4\nu, \qquad s=f-1=\epsilon^2/4\nu.$$

But $r^2-\nu s^2$ must be a square. Hence $s=0$, $U=0$, $E=L=J$, so that the algebra is a field. The case $\sigma\neq-1$ is excluded since then

$$\epsilon=\frac{2r\sigma}{\sigma+1}, \quad t=-\sigma^2-\frac{1}{\nu}\left(\frac{2r}{\sigma+1}\right)^2, \quad f=\frac{(\sigma-1)^2}{4}+\frac{r^2}{\nu(\sigma+1)^2}, \quad s=\sigma+f,$$

$$r^2-\nu s^2=r^2-\nu\left\{\frac{(\sigma+1)^2}{4}+\frac{r^2}{\nu(\sigma+1)^2}\right\}^2=-\nu\left\{\frac{(\sigma+1)^2}{4}-\frac{r^2}{\nu(\sigma+1)^2}\right\}^2.$$

If we proceed without the specialization $c_1=0$, we find that, unless the algebra is a field, $e=-r$, $f=s+rc_1/c_2$, from which (18) follows.

7. It remains to determine which of the quaternary algebras (11) satisfying (18) and (19) are equivalent under a linear transformation of the units $1, I, J, K=IJ$. Let $(1, i, j, k)$ be equivalent to $(1, I, J, K)$. Then j and J are roots of $x^2-c_1x+c_2=0$. If $j=J$, then $i^2=j=J=I^2$, so that the algebras are identical or differ only in the sign of the parameter U. Let next $j=J'$,

so that $i^2 = J'$. We prove that, if F is a finite field, there exists an element $X + YI$ whose square is J'. By (11), the conditions are

$$(31)\quad X^2 + L(Y^2 - Y'^2) + JY'^2 = J', \qquad 2XY + U(Y^2 - Y'^2) = 0.$$

Set $U^2 = \lambda + \mu J$. Then by (14) and (18),

$$(32)\qquad L = \tfrac{1}{4}\lambda J(J - J')/c_2.$$

Set $Y^2 = e + fJ$. Then (31_2) becomes $XY = \frac{1}{2}Uf(J' - J)$. Substituting its square in (31_1) multiplied by Y^2, and applying (32), we get

$$ec_1 + fc_2 = 0, \quad -e = \tfrac{1}{4}(c_1^2 - 4c_2)[f^2\mu + \lambda f(e + c_1 f)/c_2] + e^2 + efc_1 + f^2c_2.$$

But $e \neq 0$, since $Y = 0$ requires $X^2 = J'$, while J is a not-square. Hence

$$(33)\quad f = -ec_1/c_2, \quad e\omega = -4c_2^3, \quad \omega \equiv (c_1^2 - 4c_2)\{\mu c_1^2 c_2 - \lambda c_1(c_2 - c_1^2)\} + 4c_2^3.$$

For $U^2 = \lambda + \mu J$, (19) becomes, on multiplication by the not-square $c_1^2 - 4c_2$,

$$(34)\quad \lambda^2(c_1^2 - 4c_2)^2 - 8\lambda c_1 c_2(c_1^2 - 4c_2) - 16c_2^2(c_1^2 - 4c_2)(\mu - 1) = \text{not-square}.$$

We may now show that $\omega \neq 0$. As this is obvious when $c_1 = 0$, we set $c_1 \neq 0$. Then if $\omega = 0$, we have $c_2(c_1^2 - 4c_2)\mu$ expressed as a linear function of λ. Substituting this value in (34), we obtain for the left member

$$\{\lambda(c_1^2 - 4c_2) - 4(2c_2^2/c_1 - c_1c_2)\}^2.$$

Thus e and f are uniquely determined in F by (33). Then

$$(35)\qquad i = \tfrac{1}{2}UAc_1(J' - J)/J + AJI, \qquad A^2 \equiv -e/c_2.$$

If u is the parameter of the algebra $(1, i, j, k)$, $u(j - j') \equiv u(J' - J)$ is the coefficient of i in ik. We readily find that

$$(36)\qquad u = c_2AU(c_1J^{-2} - J^{-1}) = AUJ'^3/c_2.$$

We proceed to prove that, if $U \neq 0$, the new algebra, with the parameter u, is distinct from the earlier algebras, with the parameters U and $-U$. Suppose that $u(j) = \pm U(J)$. Set $u' = u(J') \equiv u(j)$. Then must $u' \equiv \pm U$, as functions of J. Hence, by (36), $A'U'J'^3 = \pm c_2U$. Then $AUJ'^3 = \pm c_2U'$. Forming the product, we get $AA'c_2 = 1$, since $JJ' = c_2$ and $U \neq 0$. But $A^2 = A'^2 = -e/c_2$. Hence $e^2 = 1$. If $e = -1$, then $A' = -A$; for, if A be a mark of the field F, c_2 would be a square. Then $AA'c_2 = 1$ and $A^2 = 1/c_2$ are contradictory. Hence $e = 1$. Thus $U'^2J^6 + c_2^3U^2 = 0$. Express the left

member as a linear function of J by applying $U^2 = \lambda + \mu J$, $J^2 - c_1 J + c_2 = 0$. Aside from the non-vanishing factor $c_1^2 - c_2$, the coefficient of J is

$$\mu c_2(c_1^2 - 2c_2) + \lambda c_1(c_1^2 - 3c_2) = 0. \tag{37}$$

The constant term equals $-c_1 c_2$ times the preceding expression. By (33), $\omega = -4c_2^3$, and $c_1 \neq 0$. We thus have two linear equations in λ and μ. We get

$$\lambda = \frac{-4c_2(c_1^2 - 2c_2)}{c_1(c_1^2 - 4c_2)}, \qquad \mu = \frac{4(c_1^2 - 3c_2)}{c_1^2 - 4c_2}, \tag{38}$$

where $c_1^2 - 4c_2$ is a not-square, being the discriminant of (6) with $n = 2$. The expression (19) is now found to reduce to

$$64c_2^4 \div c_1^2(c_1^2 - 4c_2),$$

a not-square in F. We have now proved the following

THEOREM. *Consider the set of all quaternary algebras* (11) *satisfying conditions* (18) *and* (19), *where J is a root of a fixed quadratic equation irreducible in the field F. When F is a finite field, the algebras with $U \neq 0$ are equivalent in sets of four, and the identity is the only transformation of such an algebra into itself.*

The algebra (2), *given by $U = 0$, is not equivalent to an algebra with $U \neq 0$, and admits exactly four transformations into itself, viz.,*

$$i = \pm I; \qquad i = \rho I, \qquad \text{where} \qquad \rho^2 = J^2/c_2.$$

THE UNIVERSITY OF CHICAGO,
June, 1906.

ON THE ORDER OF LINEAR HOMOGENEOUS GROUPS*

(SUPPLEMENT)

BY

H. F. BLICHFELDT

§ 1. The author has given a superior limit to the value of a prime p which may divide the order of a finite, primitive group of linear homogeneous substitutions, of determinants unity, in n variables, namely $p \leqq (n-1)(2n+1)$ (with reductions for the cases $n \leqq 6$).† He has also proved that, if such a group G contains a substitution S of variety $m (\leqq n)$ and of order $p^{a+c} \geqq mp^c$, then will G contain an invariant subgroup H (or be itself such a group), which possesses the property that, if V be a substitution of G, and T one of H, then is $(V)_p \equiv (VT)_p \pmod p$.‡

The groups not containing H being by far the most difficult to determine, at least when n is small, the author proves, in the present paper, a theorem supplementing the one just stated, giving a lower general limit to the highest power of p which may divide the order of a group G not containing H, than flows from the theorem stated. The paper also gives further reductions to the general formula for the limit to p, as well as to the special cases when $n \leqq 6$.

§ 2. THEOREM 14. *If a group G in n variables has an abelian subgroup K of order $p^a \geqq p^n$, then will G have an invariant subgroup H containing a subgroup of K of order p^{a-n+1}. If S is any substitution of H, and V any of G, then is $(V)_p \equiv (VS)_p \pmod p$.*

Let K be written in canonical form, and let its substitutions be $I = S_0$, S_i $(i = 1, 2, \cdots, p^a - 1)$, with the multipliers

$$(1, 1, 1, \cdots, 1), (\theta_{1,i}, \theta_{1,i}\theta_{2,i}, \theta_{1,i}\theta_{3,i}, \cdots, \theta_{1,i}\theta_{n,i}).$$

To the determinant (6) of *L-G* II will correspond the following matrix of p^a rows and $1 + m$ columns, where m corresponds to "variety" of theorem 10:

* Presented to the Society (Chicago), April 14, 1906. Received for publication February 13, 1906.

† *On the Order of Linear Homogeneous Groups*, Transactions, vol. 4 (1903), pp. 387–397.

‡ *On the Order of Linear Homogeneous Groups* (second paper), ibid., vol. 5 (1904), pp. 310–325. The theorem, stated on page 315 of the paper, is numbered 10. These two articles will be referred to by *L-G* I and *L-G* II, respectively.

$$1) \qquad \begin{vmatrix} (V) & 1 & 1 & \cdots & 1 \\ (VS_1)\theta_{1,1}^{-1} & 1 & \theta_{2,1} & \cdots & \theta_{m,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (VS_i)\theta_{1,i}^{-1} & 1 & \theta_{2,i} & \cdots & \theta_{m,i} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Now, to this matrix* may be added p^{a-m+1} rows of the form

$$(2) \qquad (VS_i) - (V) + (1-\theta)X_i, 0, 0, \cdots, 0,$$

where θ is a root of $\rho^{p^a} - 1 = 0$, and where X_i is a linear function of the quantities $(V), (VS_1), \cdots$, with coefficients that are integral functions of θ, the numerical coefficients entering being integers or fractions whose denominators are prime to p. To prove this proposition we shall assume it true for all possible matrices of similar form, but containing fewer columns. Under this assumption, we may add to the matrix (1) $p^{a-m+2} - 1$ rows of the form

$$(VS_i) - (V) + (1-\theta)X_i, 0, 0, \cdots, 0, Y_i,$$

where the quantities Y_i are integral functions of θ with numerical coefficients which are integers or fractions whose denominators are prime to p. Let us suppose that i can take any one of the values $2, 3, \cdots, p^{a-m+2}$.

It is readily proved that Y_i may be written in the form

$$Y_i = a_i(1-\theta)^{\alpha_i} + b_i(1-\theta)^{\beta_i} + c_i(1-\theta)^{\gamma_i} + \cdots \quad (\alpha_i < \beta_i < \gamma_i < \cdots),$$

where a_i is one of the numbers $1, 2, \cdots, p-1$, while $b_i, c_i, \cdots$ are integers or fractions whose denominators are prime to p. We shall suppose the rows arranged in such a way that

$$\alpha_2 = \alpha_3 = \cdots = \alpha_{k+1} < \alpha_j \quad (j = k+2, k+3, \cdots, p^{a-m+2}).$$

First, if $k \leqq (p-1)p^{a-m+1}$, we obtain $p^{a-m+2} - (k+1) \geqq p^{a-m+1} - 1$ rows of the form

$$(VS_j) - (V) + (1-\theta)X_j - \frac{Y_j}{Y_2}[(VS_2) - (V) + (1-\theta)X_2], 0, 0, \cdots, 0;$$

i. e., of the form

$$(VS_j) - (V) + (1-\theta)X_j', 0, 0, \cdots, 0.$$

Here the quantities X_j' are easily proved to be of the same general form as the quantities X_i, and hence the proposition assumed true for a matrix with m columns, is proved true for one of $m+1$ columns.

Next, if $k > (p-1)p^{a-m+1}$, we proceed thus: The numbers $a_2, a_3, \cdots, a_{k+1}$, taking the values $1, 2, \cdots, p-1$ only, may be separated into lots, among

* It is to be noticed that no two of the last m columns are identical.

which there must be one of $l \geqq k/(p-1) \geqq p^{a-m+1}$ numbers (say $a_2, a_3, \cdots a_{l+1}$), having one and the same value a_2. Then we derive $l-1 \geqq p^{a-m+1}-1$ rows of the form

$$(VS_i) - (V) + (1-\theta)X_i - \frac{Y_1}{Y_2}[(VS_2) - (V) + (1-\theta)X_2],\ 0,\ 0,\ \cdots,\ 0;$$

i. e., of the form

$$(VS_i) - (VS_2) + (1-\theta)X_i',\ 0,\ 0,\ \cdots,\ 0.$$

But, if $l-1$ such rows result, it is readily proved that we could obtain $l-1$ rows of the form

$$(VS_2^{-1}S_i) - (V) + (1-\theta)X_i'',\ 0,\ 0,\ \cdots,\ 0,$$

and the proposition is fully proved.

§ 3. Let us now consider the matrix (1). We may add the $p^{a-m+1}-1$ rows (2) to the p^a rows in (1). Provided that at least one of the determinants of m^2 elements, contained in the matrix obtained by erasing the first column of (1), does not vanish, we get, then, p^{a-m+1} equations of the form (including the identity when $S_i = I$):

$$(VS_i) - (V) + (1-\theta)X_i = 0.$$

The quantities X_i being integral functions of certain roots of unity, and containing no numerical coefficients whose denominators are multiples of p, the equations obtained may be written

$$(VS_i)_p - (V)_p \equiv 0 \pmod{p}.$$

From these it follows that the group G considered has an invariant subgroup H containing a subgroup of K of order p^{a-m+1}, and theorem 14 is proved (cf. the arguments in the proof of theorem 10 in *L-G* II).

§ 4. There remains to prove that there is at least one non-vanishing determinant of m^2 elements in the matrix

$$(3) \qquad \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \theta_{2,1} & \cdots & \theta_{m,1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \theta_{2,i} & \cdots & \theta_{m,i} \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

Since no two of the columns are identical, there must be at least one root $\phi \neq 1$ in the rth column, $r \neq 1$, and, therefore, also one $\phi^{p^b} = \rho \neq 1$, $\rho^p = 1$, ρ being one of the multipliers of the substitution $S_h \theta_{1,h}^{-1}$, say. If we exhibit the abelian group K in the form

$$K \equiv A(1 + S_h + S_h^2 + \cdots + S_h^{p-1}) + B(1 + S_h + S_h^2 + \cdots + S_h^{p-1}) + \cdots,$$

$A, B, \cdots$ being substitutions of K not identical with powers of S_k, it becomes evident that the sum of all the roots in the column considered can be written so as to have $1 + \rho + \rho^2 + \cdots + \rho^{p-1}$ as a factor and is therefore zero. Accordingly, by adding all the rows of (3) together, we get a row of the form

$$p^a,\ 0,\ 0,\ \cdots,\ 0.$$

Moreover, it is clear that we can multiply the rows by such roots of unity that their sum is

$$0,\ p^a,\ 0,\ \cdots,\ 0;$$

etc. There results a determinant whose value is p^{ma}, and theorem 14 is fully proved.

§ 5. It now follows that, if the order of a collineation group G in n variables is divisible by p^{n+n_p}, when p is a prime, and n_p denotes the highest power of p that divides $n!$, then will G contain a self-conjugate subgroup H, or be itself such a group. For, a group of order p^k can be written in monomial form (theorem 9, *L-G* II). The group of order p^{n+n_p} contained in G will accordingly have an abelian subgroup of order p^n at least, which subgroup, when written as a linear homogeneous group, will be of order p^{n+l}, if it contains a group of similarity-substitutions of order p^l.

COROLLARY. *The factor of the order of a collineation group G (not containing H), which is the product of primes each $\leqq n$, must divide $n!\,(2\cdot 3\cdots p)^{n-1}$, where $2, 3, \cdots, p$ denote all the different primes $\leqq n$.*

§ 6. For the purpose of lowering the limit of $p\,(p > n)$, we shall consider the case of a group G containing a substitution S of order p and of variety m, but no substitution of order $pk\,(k > 1)$ unless such a substitution is the product of one of order p and a similarity substitution. Let V be any substitution of G. In the series of weights,

$$(V),\ (VS),\ \cdots,\ (VS^{p-1}),$$

write $+1$, 0 or -1 for every root of unity of an index prime to p, according to the scheme explained in § 3 of *L-G* I. We shall, however, leave unchanged the roots of index p. The resulting expressions will be indicated by $[V], [VS], \cdots$. Then every $[VS^a]$ will be an integer lying between $-n$ and $+n$, inclusive, if the order of VS^a is prime to p; otherwise $[VS^a]$ will be the sum of n pth roots of unity,* the negative of such a sum, or 0, depending on the value (1, -1 or 0) alloted to the multipliers of the similarity substitution $\{VS^a\}^p$. Let w denote the number of weights $[VS^a]$ which are

* No primitive group in n variables can have a substitution of order p^2, if $p > n$. See Cor. 1, page 316, of *L-G* II.

sums or the negatives of sums of n pth roots of unity, and let m' be the least number of different, primitive pth roots contained in any one of these weights ($1 \leqq m' \leqq n$). Then we shall prove that $w \geqq p + 1 - m - m'$.

Let θ be a primitive pth root of unity, and let $\theta^{\alpha_1}, \theta^{\alpha_2}, \cdots, \theta^{\alpha_{p-m}}$ be the $p - m$ roots whose reciprocals are not found among the multipliers of S. Then we have

$$\sum_{r=0}^{p-1} [VS^r]\,\theta^{\alpha_i r} = 0 \qquad (i = 1, 2, \cdots, p-m).$$

When θ is considered as a variable, the left hand members will not necessarily vanish, but we will have identities in θ of the form (by KRONECKER's theorem; see § 3 of *L-G* I):

$$\sum_{r=0}^{p-1} [VS^r]\,\theta^{\alpha_i r} \equiv (1 + \theta + \theta^2 + \cdots + \theta^{p-1})\,X_i,$$

the quantities X_i being integral functions of θ with integral coefficients. Let us operate upon these identities in turn by ϑ, ϑ^2, ϑ^3, $\cdots (\vartheta = \theta\,\partial/\partial\theta)$, and put 1 for θ after differentiation. If we indicate $\{\vartheta^a [VS^r]\}_{\theta=1}$ by A_r^a, we obtain the congruences (mod p)

$$\begin{aligned}
&\textstyle\sum A_r^1 + \alpha_i \sum A_r^0 r \equiv 0,\\
&\textstyle\sum A_r^2 + 2\alpha_i \sum A_r^1 r + \alpha_i^2 \sum A_r^0 r^2 \equiv 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\textstyle\sum A_r^{p-2} + \cdots + \alpha_i^{p-2} \sum A_r^0 r^{p-2} \equiv 0 \qquad (i = 1, 2, \cdots, p-m),
\end{aligned}$$

from which we derive the following:

$$\begin{aligned}
&\textstyle\sum A_r^1 \equiv 0,\ \sum A_r^0 r \equiv 0;\ \sum A_r^2 \equiv 0,\ \sum A_r^1 r \equiv 0,\ \sum A_r^0 r^2 \equiv 0;\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\textstyle\sum A_r^{p-m-1} \equiv 0,\ \sum A_r^{p-m-2} r \equiv 0,\ \cdots,\ \sum A_r^0 r^{p-m-1} \equiv 0 \pmod p;
\end{aligned}$$

the summation in each case extending over the p values of r.

Suppose that $w < p + 1 - m - m'$. Then is every $A_r^a = 0$ $(a > 0)$, except for at most $p - m - m'$ subscripts r. For these subscripts we have

$$\textstyle\sum A_r^1 \equiv 0,\ \sum A_r^1 r \equiv 0,\ \cdots,\ \sum A_r^1 r^{p-m-2} \equiv 0,$$

and therefore every $A_r^1 \equiv 0$, if $p - m - 1 \geqq p - m - m'$. We also find that every $A_r^2 \equiv 0$, if $p - m - 2 \geqq p - m - m'$, etc. Finally every $A_r^{m'} \equiv 0$.

Now, one of the quantities $[VS^r]$ had just m' distinct primitive pth roots of unity, so that it may be written

$$[VS^r] = \pm\alpha \pm \beta\theta_1 \pm \gamma\theta_2 + \cdots \pm \kappa\theta_{m'} \quad (\alpha, \beta, \cdots \text{being positive integers} < n < p).$$

But, from $A_r^1 \equiv 0$, $A_r^2 \equiv 0$, $\cdots$, $A_r^{m'} \equiv 0$ follow $\beta \equiv 0$, $\gamma \equiv 0$, $\cdots$, and

therefore $\beta = \gamma = \cdots = 0$, an absurdity. Hence the proposition:

$$w \geqq p + 1 - m - m'.$$

Now, according to §§ 4–5 of *L-G* I, we have the congruence

$$A_r^0 \equiv ar^{m-1} + br^{m-2} + \cdots = f(r) \pmod{p},$$

where $a, b, \cdots$ are certain integers. The numbers A_r^0 are integers all lying between $-n$ and $+n$ inclusive, and, by the proposition just proved, at least $p + 1 - m - m'$ of them are either $+n$ or $-n$. However, at most $2(m-1)$ of the remainders of $f(r) \pmod{p}$ can be $+n$ or $-n$, unless $f(r)$ is merely a constant. This it could not be in all cases, as then, for every V of G, we would have $(V)_p \equiv (VS)_p \pmod{p}$, and G would have an invariant subgroup H, which would be of order p^l and would therefore be abelian. In such a case G could not be primitive (theorems II and III, *L-G* I). Thus,

$$2(m-1) \geqq p + 1 - m - m' \quad \text{or} \quad p \leqq 3m + m' - 3 \leqq 3m + n - 3.$$

We have now proved

THEOREM 15. *If a primitive collineation-group G in n variables has a substitution of order p $(p > n)$ and of variety m $(m \leqq n)$, but none of order pk, then is $p \leqq 3m + n - 3$.*

If, in such a group, there is a substitution of order pk, which is the product of one (S) of order p and one of order k, then can the variety of S be $n-1$ at most, unless there is an invariant subgroup H (theorem 11, *L-G* II). The number p can then not exceed $(n-2)(2n+1)$.

§ 7. If $n = 4$, the theorem VI of *L-G* I states that $p \leqq 13$. For $p = 13$ we can, however, find no function $ar^3 + br^2 + cr + d \not\equiv d$, all of whose remainders (mod 13) lie between -4 and $+4$ inclusive, and $13 + 1 - 8 = 6$ of which have the values $+4$ or -4. Accordingly, by what precedes, a primitive group in 4 variables can have no substitution S of order 13 unless it has one of order $13k$, expressible as a product SR ($S^{13} = 1$, $R^k = 1$; R and S permutable; R not a similarity-substitution), whose weight is $\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4$, where $\alpha_i^{13} = 1$, $\beta_i^k = 1$. In this case we form the determinant corresponding to (3) of *L-G* I, with (T), (R^aST), (R^bS^2T), (R^cS^3T), (R^dS^rT) for the elements of the first column. We will choose a, b, c such that the resulting equation can be solved for (R^dS^rT), and then replace all the 13th roots of unity by 1. Now, a primitive group G in 4 variables could not have an invariant subgroup H of such a nature that, if V and S be any substitutions of G and H respectively, $(V)_{13} \equiv (VS)_{13} \pmod{13}$. It follows that (R^dS^rT) is not expressible in the form $Ar + B$ (A and B being independent of r), since in that event there would be just such a subgroup H. Excluding, therefore,

this possibility, we find readily that three of the roots β_i are equal to each other, say $\beta_2 = \beta_3 = \beta_4$, and then that

$$(R^s S^r T)_{13} \equiv \beta_1^s A + \beta_2^s (Br^2 + Cr + D) \pmod{13},$$

the quantities A, B, C, D being independent of r and s.

If $A \equiv 0$, then is $(R^s S^r T)_{13} \equiv (S^r T)_{13} \beta_2^s \pmod{13}$; or, putting $T = S^{-r}$, $(R^r)_{13} \equiv 4\beta_2^s \pmod{13}$. But this is an impossible congruence for $s = 1$. Hence, $A \not\equiv 0$, and we have

$$(R^s S^r T)_{13} \beta_2^{-s} \equiv \left(\frac{\beta_1}{\beta_2}\right)^s A + Br^2 + Cr + D \pmod{13}.$$

The values $-1, 0, 1$ may now be assigned to the roots of unity involved, in accordance with the scheme of § 3, *L-G* I; and, as $\beta_1 \neq \beta_2$, we have a function

$$B_1 r^2 + C_1 r + D_1 + A' \not\equiv D_1 + A' \pmod{13},$$

B_1, C_1, D_1 and A' being integers, the last of which is capable of taking, at our will, at least two different values. All the remainders (mod 13) of this function should lie between -4 and $+4$ inclusive. But such a function does not exist.*

In a similar manner we may deal with the cases $n = 5$, $p = 17$ or 19; $n = 6$, $p = 23$.† The results are expressed in the following

THEOREM 16. *The primes which may divide the orders of the primitive collineation-groups in* 4, 5 *and* 6 *variables are, respectively, not greater than* 11, 13 *and* 19.

BERLIN,
January, 1906.

* In the article written by the author on quaternary groups, Mathematische Annalen, vol. 60 (1905), pp. 204–231, it is proved, in a different manner, that the primitive collineation-groups in 4 variables can have no substitutions of order 11 or 13.

† The cases $n = 6$, $p = 17$ or 19 have not been examined by the author. It is very likely that we may, by the process given above, throw out at least 19.

ON AUTOMORPHIC GROUPS WHOSE COEFFICIENTS ARE INTEGERS IN A QUADRATIC FIELD*

BY

J. I. HUTCHINSON

In a memoir by X. STOUFF † an interesting method is given for the determination of certain groups of linear transformations of a single variable, the coefficients being of the form $\sum_\sigma \alpha_\sigma (j^\sigma + j^{-\sigma})$ in which σ and α_σ are integers and j is a primitive pth root of unity. This method imposes conditions on the α_σ by reason of which all are linearly expressible in terms of four; these four are connected by a quadratic relation, the condition for determinant 1. The method employed by STOUFF is capable of defining only a very restricted class of groups and the explicit forms of these are deduced only in a few individual cases. The class may be greatly enlarged by including all groups whose coefficients are linear functions of four variable integers subject to a quadratic condition. In the following paper I consider groups of this type whose coefficients are of the form $\alpha + \alpha'\lambda$ in which λ is a root of the equation

$$\lambda^2 - m\lambda + n = 0, \tag{1}$$

and α, α', m, n are integers. The coefficients of any transformation of the group, $\zeta' = (A\zeta + B)/(C\zeta + D)$, form the determinant

$$\begin{vmatrix} \alpha + \alpha'\lambda & \beta + \beta'\lambda \\ \gamma + \gamma'\lambda & \delta + \delta'\lambda \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}. \tag{2}$$

This determinant is assumed to be unimodular and hence,

$$\alpha\delta - \beta\gamma - n(\alpha'\delta' - \beta'\gamma') = 1, \tag{3}$$

$$\alpha\delta' - \alpha'\delta - \beta\gamma' - \beta'\gamma + m(\alpha'\delta' - \beta'\gamma') = 0. \tag{4}$$

I further suppose that the integers γ, γ', δ, δ' are expressible in terms of α, α', β, β' in the form

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† *Sur certains groupes fuchsiens formés avec les racines d'equations binomes*, Annales de Toulouse, vol. 4 (1890), P.

$$
\begin{aligned}
p\gamma &= a_1\alpha + b_1\alpha' + c_1\beta + d_1\beta', \\
p\gamma' &= a_2\alpha + b_2\alpha' + c_2\beta + d_2\beta', \\
p\delta &= a_3\alpha + b_3\alpha' + c_3\beta + d_3\beta', \\
p\delta' &= a_4\alpha + b_4\alpha' + c_4\beta + d_4\beta',
\end{aligned} \tag{5}
$$

in which $p, a_1, \cdots$ are integers. Let these expressions be substituted in (4) and the coefficients of $\alpha^2, \alpha\alpha', \alpha'^2, \cdots$ be equated to zero. This leads to the following conditions:

$$
\begin{gathered}
d_1 = mc_1, \quad c_2 = 0, \quad d_2 = -c_1, \quad b_3 = ma_3, \quad c_3 = b_2 - ma_2, \\
d_3 = b_1 + mb_2 - m(a_1 + ma_2), \quad a_4 = 0, \quad b_4 = -a_3, \quad c_4 = a_2, \quad d_4 = a_1 + ma_2.
\end{gathered} \tag{6}
$$

In order that identity may belong to the group the additional conditions $a_1 = a_2 = 0$, $a_3 = p$ must be imposed. The equations (5) now take the simpler form

$$
\begin{aligned}
p\gamma &= b_1\alpha' + c_1(\beta + m\beta'), \\
p\gamma' &= b_2\alpha' - c_1\beta', \\
p\delta &= p(\alpha + m\alpha') + b_2\beta + (b_1 + mb_2)\beta', \\
\delta' &= -\alpha'.
\end{aligned} \tag{7}
$$

It may readily be verified that the inverse of a given transformation, and the product of any two of the form (7) have coefficients which are also of this form. Hence,

The totality of transformations whose coefficients satisfy conditions (3) *and* (7) *form a group.*

This group will be denoted by g. It remains to show that g is properly discontinuous in the plane of the variable ζ. For this purpose it is sufficient to prove that, if the coefficients A, B, C, D are restricted in numerical value, there are only a finite number of values of $\alpha, \alpha', \beta, \beta'$.* Assuming then the inequalities

$$
|A| < F_1, \qquad |B| < F_2, \qquad |C| < F_3, \qquad |D| < F_4,
$$

in which $F_1, \cdots, F_4$ are any finite positive numbers, let $f_1, \cdots, f_4$ be defined by the equations

$$
\begin{aligned}
f_1 &= pc_1(m^2 - m\lambda - 2n) - (b_1^2 + mb_1b_2 + nb_2^2), \\
f_2 &= -\lambda c_1(b_1 + b_2\lambda),
\end{aligned}
$$

*Cf. STOUFF, loc. cit., p. 5.

$$f_3 = \lambda p\,[b_1 + (m-\lambda)b_2],$$

$$f_4 = \lambda p c_1(2\lambda - m).$$

When these expressions are substituted in the inequality

$$|f_1 A + f_2 B + f_3 C + f_4 D| < \sum_{i=1}^{4} |f_i| F_i \tag{8}$$

it reduces to

$$|(f_1 + f_4)\alpha| < \Sigma |f_i| F_i.$$

As the right member is a fixed positive number, and α is restricted to integer values, it follows that α can take only a finite number of values provided that the expression

$$f_1 + f_4 = pc_1(4n - m^2) + b_1^2 + mb_1b_2 + nb_2^2 \tag{9}$$

does not vanish. In like manner by substituting in (8) the expressions

$$-f_1 = f_4 = pc_1(2\lambda - m), \qquad f_2 = -c_1(b_1 + \lambda b_2),$$

$$f_3 = p\,[b_1 + (m-\lambda)b_2],$$

we obtain the inequality

$$|Q\alpha'| < \Sigma |f_i| F_i$$

in which Q denotes the right member of (9). Again, by taking

$$-f_1 = f_4 = p\lambda(b_1 + \lambda b_2), \qquad f_3 = p^2\lambda(2\lambda - m),$$

$$f_2 = pc_1(\lambda - m)(2\lambda - m) - (b_1 + mb_2)(b_1 + \lambda b_2),$$

we deduce

$$|Q\beta| < \Sigma |f_i| F_i;$$

and finally, with the expressions

$$-f_1 = f_4 = p(b_1 + \lambda b_2), \qquad f_3 = p^2(2\lambda - m),$$

$$f_2 = -b_2(b_1 + \lambda b_2) - c_1 p(2\lambda - m),$$

we obtain

$$|Q\beta'| < \Sigma |f_i| F_i.$$

Hence, *if the integer* $Q = pc_1(4n - m^2) + b_1^2 + mb_1b_2 + nb_2^2$ *does not vanish, there are only a finite number of integer values which* α, α', β, β' *can take when the numerical values of* A, B, C, D *are restricted, and the group is therefore properly discontinuous in the complex plane.*

The preceding demonstration is necessary only in case λ is real. If λ is imaginary, the group is evidently discontinuous since no complex integer of the form $\beta + \lambda\beta'$ can be infinitesimal.

The group g may be enlarged by including every substitution V whose square belongs to g.* Assuming that the coefficients of V satisfy conditions (6) only, while those of V^2 are of the form (7), we obtain for V a substitution of form (2) whose coefficients are subject to the conditions

$$
\begin{aligned}
p\gamma &= -(mb_1 + 2nb_2)(2\alpha + m\alpha')\Delta^{-1} - c_1(\beta + m\beta'),\\
p\gamma' &= (2b_1 + mb_2)(2\alpha + m\alpha')\Delta^{-1} + c_1\beta',\\
p\delta &= -p(\alpha + m\alpha') - [m(2b_1 + mb_2)\beta + \{m^2b_1 + m(m^2 - 2n)b_2\}\beta']\Delta^{-1},\\
p\delta' &= p\alpha' + [(4b_1 + 2mb_2)\beta + \{2mb_1 + 2(m^2 - 2n)b_2\}\beta']\Delta^{-1},\\
\Delta &= m^2 - 4n.
\end{aligned} \tag{10}
$$

It can be verified that the product of any two substitutions V belongs to g and hence, *the totality of transformations whose coefficients satisfy conditions* (7) *or* (10) *form a group.* This enlarged group will be denoted by G. The transformations of G will be spoken of as of the first or second type and will be denoted by v or V according as they satisfy conditions (7) or (10) respectively.

Since $A + D$ is an integer for substitutions v, it follows that elliptic substitutions v can be of periods 2 and 3 only, and hence elliptic substitutions of the second type cannot have other periods than 2, 4, and 6. But for the substitutions V we have $A + D = (2\lambda - m)I = I\sqrt{\Delta}$ in which I is an integer. Those of period 4 can occur only when $\Delta = 2$, and those of period 6 only when $\Delta = 3$. But if $\Delta = \epsilon(\epsilon = 2, 3)$, we have $m^2 = 4n + \epsilon$ which is impossible since ϵ is not a quadratic residue of 4. Hence, *the substitutions V are either of period* 2, *or hyperbolic.*

In order that the group G may be extended by the reflection $\zeta' = -\bar{\zeta}$ on the imaginary axis it is necessary and sufficient that with every substitution (2) the substitution $\left|\begin{smallmatrix} A & -B \\ -C & D \end{smallmatrix}\right|$ shall also be included in the group (λ being real). This is possible only when $b_1 = b_2 = 0$. Write $c_1 = pq$ and $m - \lambda = \lambda'$. Then, *the most general group G which can be extended by reflection on the imaginary axis consists of the transformations*

$$
\text{(I)}\quad \begin{vmatrix} \alpha + \alpha'\lambda & \beta + \beta'\lambda \\ q(\beta + \beta'\lambda') & \alpha + \alpha'\lambda' \end{vmatrix}, \qquad \text{(II)}\quad \begin{vmatrix} \alpha + \alpha'\lambda & \beta + \beta'\lambda \\ -q(\beta + \beta'\lambda') & -(\alpha + \lambda'\alpha') \end{vmatrix},
$$

the coefficients of which are subject to the condition

$$
\alpha^2 + m\alpha\alpha' + n\alpha'^2 - q(\beta^2 + m\beta\beta' + n\beta'^2) = \pm 1. \tag{11}
$$

* The question naturally arises as to whether it would be possible to extend g by a substitution V whose nth power ($n > 2$) and no lower power is contained in g. That this is not possible in general is shown by proving, as may readily be done, the impossibility of such an extension in case of the particular groups for which $b_1 = b_2 = 0$.

The plus and minus signs correspond to (I) and (II) respectively. This group will be denoted by $G_{\{q,\lambda\}}$ or more briefly by $\{q, \lambda\}$, while the substitutions (I) form a subgroup $g_{\{q,\lambda\}}$.

The determinant for (I) may be written

$$(2\alpha + m\alpha')^2 - \Delta\alpha'^2 - q[(2\beta + m\beta')^2 - \Delta\beta'^2] = 4.$$

Since the sum of the diagonal coefficients is $A + D = 2\alpha + m\alpha'$ it follows that elliptic substitutions are subject to the condition

$$-\Delta\alpha'^2 - q[(2\beta + m\beta')^2 - \Delta\beta'^2] = \epsilon, \tag{12}$$

in which ϵ is 4 or 3 according as (I) is of period 2 or 3 respectively. If we write $2\beta + m\beta'$ in the form $M\Delta + \mu$, $0 \leqq \mu < \Delta$, it is evident that (assuming $\Delta \neq 3$) relation (12) is impossible unless μ satisfies the congruence $-q\mu^2 \equiv \epsilon \pmod{\Delta}$. Hence, if r denote any quadratic residue of Δ, *the group $g_{\{q,\lambda\}}$ has no substitutions of period 2 or 3 unless the condition $-qr \equiv 4 \pmod{\Delta}$ or $-qr \equiv 3 \pmod{\Delta}$ can be satisfied by some one of the allowable values of r.*

The condition for a parabolic substitution is

$$\Delta\alpha'^2 + q(2\beta + m\beta')^2 - q\Delta\beta'^2 = 0.$$

Assume $q = cq_1q_2^2$, $\Delta = c\Delta_1\Delta_2^2$ in which q_2^2, Δ_2^2 are the highest quadratic factors in q, Δ, and c is the greatest common divisor of the remaining factors of q and Δ. After multiplying the above relation by $cq_1\Delta_1$ it takes the form

$$q_1(c\Delta_1\Delta_2\alpha')^2 + \Delta_1[cq_1q_2(2\beta + m\beta')]^2 - c(cq_1q_2\Delta_1\Delta_2\beta')^2 = 0.$$

In order that this equation in α', β, β' may have integer solutions it is necessary and sufficient that $c\Delta_1$, cq_1, $-q_1\Delta_1$ be quadratic residues of q_1, Δ_1, c respectively and do not all have the same sign.* Hence, *the group $G_{\{q,\lambda\}}$ contains parabolic substitutions when (and only when) integers z, z', z'' can be found to satisfy the congruences*

$$z^2 \equiv c\Delta_1 \pmod{q_1},$$
$$z'^2 \equiv cq_1 \pmod{\Delta_1},$$
$$z''^2 \equiv -q_1\Delta_1 \pmod{c},$$

in which $q_1, \Delta_1, -c$ are not all of the same sign.

The infinity of groups $\{q, \lambda\}$ obtained by giving q different integer values are not all distinct when regarded as abstract groups. For, let $\{q, \lambda\}$ be transformed by means of

$$T = \begin{vmatrix} t + t'\lambda & 0 \\ 0 & 1 \end{vmatrix},$$

*See DIRICHLET, *Zahlentheorie*, p. 432.

in which t, t' are integers. The transformed of the operations (I) are

$$\begin{vmatrix} \alpha+\alpha'\lambda & b+b'\lambda \\ \frac{q}{\tau}(b+b'\lambda') & \alpha+\alpha'\lambda' \end{vmatrix}$$

in which

(13) $b=t\beta-nt'\beta', \qquad b'=t\beta'+t'\beta+mt'\beta', \qquad \tau=t^2+mtt'+nt'^2.$

On account of (11) we have also the restriction

(14) $$(\alpha+\alpha'\lambda)(\alpha+\alpha'\lambda')-\frac{q}{\tau}(b+b'\lambda)(b+b'\lambda')=1.$$

A corresponding result is obtained by transforming (II). Hence, *the transformed of* $\{q,\lambda\}$ *by* T *is the group* $\{q/\tau,\lambda\}$. In particular, whenever m and n are such that -1 can be represented by τ then $\{q,\lambda\}$ is isomorphic with $\{-q,\lambda\}$.

Suppose $q=\tau q'$. To each pair of values of β, β' satisfying (11) corresponds one pair of values of b, b' satisfying (14). But from the equations

$$\beta=\tau^{-1}[(t+mt')b+nt'b'], \qquad \beta'=\tau^{-1}[-t'b+tb'],$$

derived from (13), it is seen that to each pair of integer values of b, b' satisfying (14) do not always correspond integer values of β, β'. Accordingly, *when* $q=\tau q'$ *the group* $\{q,\lambda\}$ *can be transformed into a subgroup of* $\{q',\lambda\}$. In particular, for every number q which can be represented by τ the group $\{q,\lambda\}$ can be transformed into a subgroup of $\{1,\lambda\}$.

This result gives a method of representing an important class of subgroups of the group $\{q,\lambda\}$. Namely, *those values of* β, β' *which satisfy* (11) *and which make the expressions*

$$\tau^{-1}[(t+mt')\beta+nt'\beta'], \qquad \tau^{-1}[-t'\beta+t\beta']$$

integers determine a subgroup of $\{q,\lambda\}$.

The transformed of $\{q,\lambda\}$ by T^{-1} is $\{\tau q,\lambda\}$. If q is not an integer, suppose $q=q_1/q_2$. By choosing t, t' so that τ is divisible by q_2 the number τq reduces to an integer. Hence among the groups $\{q,\lambda\}$ it is sufficient to consider those only in which q is an integer.

We observe further that no new groups are obtained by using $\lambda'=m-\lambda$ in place of λ, since, as may readily be shown, $\{q,\lambda'\}=\{q,\lambda\}$. Moreover it is sufficient to consider only positive values of m. For, if we have $m=-m_1$, $m_1>0$, then the substitution (I) may be written

$$\begin{vmatrix} \alpha-\alpha'\lambda_1, & \beta-\lambda_1'\beta' \\ q(\beta-\lambda_1\beta'), & \alpha-\lambda_1\alpha' \end{vmatrix}, \qquad \lambda_1=\frac{m_1+\sqrt{m_1^2-4n}}{2}.$$

By replacing α', β' by $-\alpha'$, $-\beta'$ this again takes the form (I). Hence we have

$$\{q, \lambda\} = \{q, \lambda_1'\} = \{q, \lambda_1\}.$$

These considerations may readily be extended to the group G defined by (7) and (10) which we will denote for greater explicitness by the symbol $\{b_1, b_2, c_1, \lambda\}$. We obtain as result the relations,

$$\{b_1, b_2, c_1, \lambda'\} = \{-(b_1 + mb_2), b_2, c_1, \lambda\},$$

$$\{b_1, b_2, c_1, \lambda\} = \{-b_1, b_2, c_1, \lambda_1'\} = \{b_1 - mb_2, b_2, c_1, \lambda_1\}.$$

In case λ is real, the groups $\{q, \lambda\}$ are transformed by the substitution $\sqrt{q}\zeta = \eta$ into groups which reproduce the ternary form* $z_1^2 - qz_2^2 - \Delta z_3^2$. If λ be imaginary and q positive,† the substitution

$$\sqrt{q}\zeta = \frac{\eta - i}{\eta + i}$$

transforms (I) and (II) into

$$\begin{vmatrix} \frac{a + b\sqrt{q}}{2}, & \frac{c + d\sqrt{q}}{2}\sqrt{-\Delta} \\ \frac{-c + d\sqrt{q}}{2}\sqrt{-\Delta}, & \frac{a - \sqrt{q}b}{2} \end{vmatrix},$$

$$\begin{vmatrix} \frac{c - d\sqrt{q}}{2}\sqrt{-\Delta}, & \frac{-a + b\sqrt{q}}{2} \\ \frac{a + b\sqrt{q}}{2}, & \frac{c + d\sqrt{q}}{2}\sqrt{-\Delta} \end{vmatrix},$$

respectively, in which

$$a = -(2\alpha + m\alpha'),\ b = -(2\beta + m\beta'),\ c = \alpha',\ d = -\beta'.$$

These groups evidently coincide with those obtained in the case of a real λ.

In case q and Δ are both negative, the substitutions of type (I) can be reduced to the form

$$\begin{vmatrix} A & B \\ qB & A \end{vmatrix}$$

with the condition

$$A\bar{A} - qB\bar{B} = 1,$$

which can be satisfied only by a finite number of integer values of a, b, c, d since all the terms are positive. Hence the group $\{q, \lambda\}$ is finite when q and Δ are both negative.

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*See FRICKE-KLEIN, *Automorphe Functionen*, vol. I, p. 537.

†The orthogonal circle for these groups is $q\zeta\bar{\zeta} = 1$.

A CLASS OF PERIODIC SOLUTIONS OF THE PROBLEM OF THREE BODIES WITH APPLICATION TO THE LUNAR THEORY*

BY

FOREST RAY MOULTON.

1. Introduction.
2. The Differential Equations.
3. Existence of the Periodic Solutions.
4. Practical Construction of the Solutions.
5. Applications to the Lunar Theory.
6. Applications to Darwin's Periodic Orbits.

1. Introduction.

This paper treats certain periodic solutions of the problem of three finite masses which move in the same plane under their mutual attractions. The forces are supposed to vary according to the Newtonian law, and it is assumed that the bodies are spheres. The periods of the solutions are equal to the synodical periods of the bodies, and the orbits reduce to circles when the disturbing forces are put equal to zero. It follows that they are of the class called by Poincaré "*Solutions de la première sorte.*" †

In the series which have been developed for expressing the coördinates of the moon there are certain terms whose period is the same as that of its synodical revolution. In the notation of Delaunay ‡ they are the terms which involve $\sin kD$ and $\cos kD$, where k is an integer. It is shown that these terms belong to the periodic orbits treated in this paper. By putting the mass of the moon equal to zero and transforming the parameter in terms of which the expansions are made, we can compare the coefficients of the various terms with those given by Delaunay. §

As applied to the Lunar Theory the results obtained in this paper are, when the mass of the moon is neglected, coextensive with those given by Hill in his *Researches*, ‖ and by Brown in his first memoir. ¶

* Presented to the Society under a different title, September 8, 1905 and April 14, 1906. Received for publication June 26, 1906.

† *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, p. 97.

‡ Mémoires de l'Académie des Sciences, vol. 29 (1867), p. 801.

§ Loc. cit., pp. 803–924.

‖ American Journal of Mathematics, vol. 1 (1878), pp. 5–26, 129–147, 245–260

¶ American Journal of Mathematics, vol. 14 (1892), pp. 141–160.

When the masses of two of the bodies have the ratio 1 to 10, while that of the third is infinitesimal, the problem of this paper reduces to that which DARWIN treated by numerical processes.* For the sake of comparison, one of his "Satellite A" orbits and one of his "Planet A" orbits have been computed from the literal series derived below. A retrograde satellite orbit having the same sidereal period is also given. DARWIN's "Satellites B" and "Satellites C" are imaginary for small values of the disturbing forces. They belong to values of the parameter which lie beyond that for which the series have a singularity, and their coördinates, therefore, are not represented by these series. There are only three classes of orbits of the type in question, real or imaginary, in which the bodies revolve in the forward direction, and it appears, therefore, that DARWIN's search for these periodic satellite orbits was exhaustive. However, the number of retrograde orbits is the same as the number of those which revolve in the forward direction.

After having selected variables which seem most convenient for the construction of the series and for practical use, the existence of the solutions of the type in question is established. For this purpose the process of analytical continuation is employed which POINCARÉ developed in this general connection, and which he made the central idea of his researches on the figures of equilibrium of rotating fluid masses† and on the problem of three bodies.‡ After the demonstrations of the existence and the properties of the solutions have been made, the practical construction of the series is carried out along lines which have been shown to be legitimate. Instead of arranging the solutions as Fourier series, as has been customary among astronomers, power series are employed, and the work has the simplicity which is characteristic of processes involving power series. For example, every coefficient is determined by a single step and is modified by no subsequent operations. Moreover, it is possible to prove the convergency of the series within certain limits, though the precise limits where convergency ceases have not been found.

2. THE DIFFERENTIAL EQUATIONS.

Let the three masses be E, M, and S. Choosing E as the origin and letting the rectangular coördinates of M and S be x, y and X, Y respectively, we have the differential equations of motion:

$$\frac{d^2x}{dt^2} + k^2(E+M)\frac{x}{r^3} = -k^2 S\left\{\frac{x-X}{\rho^3} + \frac{X}{R^3}\right\},$$

*Acta Mathematica, vol. 21 (1897), pp. 99–242; Mathematische Annalen, vol. 51 (1898–9), pp. 523–583.

†Acta Mathematica, vol. 7 (1885), pp. 259–380.

‡Bulletin Astronomique, vol. 1 (1884), pp. 65–74; Acta Mathematica, vol. 13 (1890), pp. 8–270; *Les Méthodes Nouvelles de la Mécanique Céleste* (1892–9).

$$\frac{d^2y}{dt^2}+k^2(E+M)\frac{y}{r^3}=-k^2S\left\{\frac{y-Y}{\rho^3}+\frac{Y}{R^3}\right\},$$

$$\frac{d^2X}{dt^2}+k^2(E+S)\frac{X}{R^3}=-k^2M\left\{\frac{X-x}{\rho^3}+\frac{x}{r^3}\right\},$$

$$\frac{d^2Y}{dt^2}+k^2(E+S)\frac{Y}{R^3}=-k^2M\left\{\frac{Y-y}{\rho^3}+\frac{y}{r^3}\right\},$$

where

$$r=\sqrt{x^2+y^2},\qquad \rho=\sqrt{(x-X)^2+(y-Y)^2},\qquad R=\sqrt{X^2+Y^2}.$$

We shall now refer the motion of S to the center of gravity of E and M by the transformation

$$X=x'+\frac{M}{E+M}x,\qquad Y=y'+\frac{M}{E+M}y,\qquad r'=\sqrt{x'^2+y'^2}.$$

Then passing to polar coördinates by the transformation

$$x=r\cos v,\qquad x'=r'\cos v',$$

$$y=r\sin v,\qquad y'=r'\sin v',$$

the differential equations become

$$\frac{d^2r}{dt^2}-r\left(\frac{dv}{dt}\right)^2+\frac{k^2(E+M)}{r^2}=k^2S(E+M)\frac{\partial u}{\partial r},$$

$$r\frac{d^2v}{dt^2}+2\frac{dr}{dt}\frac{dv}{dt}=k^2S(E+M)\frac{1}{r}\frac{\partial u}{\partial v},$$

(1)

$$\frac{d^2r'}{dt^2}-r'\left(\frac{dv'}{dt}\right)^2+\frac{k^2E(E+M+S)}{(E+M)r'^2}=\frac{k^2(E+M+S)}{E+M}\frac{\partial u'}{\partial r},$$

$$r'\frac{d^2v'}{dt^2}+2\frac{dr'}{dt}\frac{dv'}{dt}=\frac{k^2(E+M+S)}{E+M}\frac{1}{r'}\frac{\partial u'}{\partial v'},$$

where

$$u=\frac{1}{E\rho}+\frac{1}{MR},$$

$$u'=E\left(\frac{1}{R}-\frac{1}{r'}\right)+\frac{M}{\rho},$$

$$\rho=r'\left[1-\frac{2E}{E+M}\frac{r}{r'}\cos(v-v')+\left(\frac{E}{E+M}\right)^2\left(\frac{r}{r'}\right)^2\right]^{\frac{1}{2}},$$

$$R=r'\left[1+\frac{2M}{E+M}\frac{r}{r'}\cos(v-v')+\left(\frac{M}{E+M}\right)^2\left(\frac{r}{r'}\right)^2\right]^{\frac{1}{2}}.$$

If $r/r' < 1$ then $1/R - 1/r'$ and $1/\rho$ can be expanded as converging power series in r/r' for all values of v and v'. Only cases in which this inequality is satisfied will be treated. In the Lunar Theory r/r' is about $\frac{1}{400}$, and in the satellite orbits computed by DARWIN, about $\frac{1}{10}$.

The series for u and u' are

$$u = \frac{1}{r'}\left\{\frac{1}{E}+\frac{1}{M}+\frac{E+M}{(E+M)^2}\left(\frac{r}{r'}\right)^2 \tfrac{1}{4}[1+3\cos 2(v-v')]\right.$$
$$+\frac{(E^2-M^2)}{(E+M)^3}\left(\frac{r}{r'}\right)^3 \tfrac{1}{8}[3\cos(v-v')+5\cos 3(v-v')]$$
$$+\frac{E^3+M^3}{(E+M)^4}\left(\frac{r}{r'}\right)^4 \tfrac{1}{64}[9+20\cos 2(v-v')+35\cos 4(v-v')]$$
$$+\frac{E^4-M^4}{(E+M)^5}\left(\frac{r}{r'}\right)^5 \tfrac{1}{128}[30\cos(v-v')+35\cos 3(v-v')+63\cos 5(v-v')]$$
$$\left.+\cdots\right\},$$

$$u' = \frac{1}{r'}\left\{M+\frac{EM}{4}\left(\frac{r}{r'}\right)^2[1+3\cos 2(v-v')]+\cdots\right\}.$$

With these values of u and u' equations (1) become

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2(E+M)}{r^2} = \frac{k^2}{2}\frac{S}{r'^3}\frac{r}{}\left\{[1+3\cos 2(v-v')]\right.$$
$$\left.+\tfrac{3}{4}\frac{(E-M)}{(E+M)}\frac{r}{r'}[3\cos(v-v')+5\cos 3(v-v')]+\cdots\right\},$$

$$r\frac{d^2v}{dt^2} + 2\frac{dr}{dt}\frac{dv}{dt} = -\frac{k^2}{2}\frac{S}{r'^3}\frac{r}{}\left\{3\sin 2(v-v')\right.$$
$$\left.+\tfrac{3}{4}\frac{(E-M)}{(E+M)}\frac{r}{r'}[\sin(v-v')+5\sin 3(v-v')]+\cdots\right\}, \qquad (2)$$

$$\frac{d^2r'}{dt^2} - r'\left(\frac{dv'}{dt}\right)^2 + \frac{k^2(E+M+S)}{r'^2} = -\frac{k^2EM(E+M+S)}{2(E+M)^2}$$
$$\times\frac{r^2}{r'^4}\left\{[1+3\cos 2(v-v')]+\cdots\right\},$$

$$r'\frac{d^2v'}{dt^2} + 2\frac{dr'}{dt}\frac{dv'}{dt} = \frac{k^2EM(E+M+S)}{2(E+M)^2}\frac{r^2}{r'^2}\left\{3\sin 2(v-v')+\cdots\right\}.$$

When the right members of these differential equations are neglected, that is, when M is supposed to revolve around E without having its motion disturbed by S and when S is supposed to revolve around the center of gravity of E and M without having its motion disturbed because of the distribution of the mass between the two separate bodies E and M, a particular solution of the equations is

$$(3)\qquad \begin{aligned} r &= a, \qquad v = \frac{k\sqrt{E+M}}{a^{\frac{3}{2}}}\, t = nt, \\ r' &= a', \qquad v' = \frac{k\sqrt{E+M+S}}{a'^{\frac{3}{2}}}\, t = n't, \end{aligned}$$

where n and n' are the angular motions of E and S respectively. In developing the periodic solutions from the differential equations including the right members, the mean angular motions will be kept the same as in the undisturbed orbits, and n and n' will be regarded as constants which are given by observation (in the applications to the Lunar Theory), or which are assumed. The masses are regarded as given, and the constants a and a' are defined by the equations

$$(4)\qquad n^2a^3 = k^2(E+M), \qquad n'^2a'^3 = k^2(E+M+S).$$

New variables p, w, p' and w' will be defined by the equations

$$(5)\qquad \begin{aligned} r &= a(1+p), \qquad v = nt + w, \\ r' &= a'(1+p'), \qquad v' = n't + w'. \end{aligned}$$

It follows that ap, w, $a'p'$ and w' are the deviations from the circular orbits of M and S due to their mutual perturbations. To reduce the differential equations to a still more convenient form we shall let

$$(6)\qquad \begin{aligned} m &= \frac{n'}{n-n'}, \text{ whence } n = \frac{n'(1+m)}{m}, \\ \tau &= (n-n')t = \frac{n'}{m}\, t, \\ \eta &= \frac{S}{E+M+S}. \end{aligned}$$

It follows from (4) and the definition of m that

$$(7)\qquad \frac{a}{a'} = \left(\frac{m}{1+m}\right)^{\frac{2}{3}}\left(\frac{E+M}{E+M+S}\right)^{\frac{1}{3}}.$$

That is, the ratio a/a' is expressible in terms of the parameter m and might be eliminated from the equations. But in the Lunar Theory a/a' is regarded as

a separate parameter and is retained explicitly. In order to keep the results in such a form that comparison may be made with the work of DELAUNAY, and at the same time to avoid the inconvenience of having power series in two parameters, we shall put in general

$$(8)\qquad \frac{E^j-(-1)^jM^j}{(E+M)^j}\left(\frac{a}{a'}\right)^{j-1}=M_{j-1}m^{2(j-1)},\qquad \left(\frac{a}{a'}\right)^j=N_jm^{2j},$$

where the M_j and N_j are determined so that (7) shall be fulfilled. The m will be regarded as a variable parameter in the argument, and the M_j and N_j as fixed constants. Therefore the m in (7) and the m in (8) belong to different problems except when they give the same value of a/a'. At the end of the work this relation will always be fulfilled, and the equations (8) amount merely to an allowable generalization of the meaning of m. The relations have been selected so that the M_j and N_j are approximately $(\frac{3}{8})^j$ in the case of the Lunar Theory, and the power of m indicates the order of the term. With these substitutions the differential equations become

$$\begin{aligned}
&\frac{d^2p}{d\tau^2}-(1+p)\left(1+m+\frac{dw}{d\tau}\right)^2+\frac{(1+m)^2}{(1+p)^2}=\frac{m^2\eta}{2}\frac{1+p}{(1+p')^3}\Big\{[1+3\cos 2(\tau+w-w')]\\
&\quad+\tfrac{3}{4}M_1m^2\frac{1+p}{1+p'}[3\cos(\tau+w-w')+5\cos 3(\tau+w-w')]\\
&\quad+\tfrac{1}{8}M_2m^4\frac{(1+p)^2}{(1+p')^2}[9+20\cos 2(\tau+w-w')+35\cos 4(\tau+w-w')]\\
&\quad+\tfrac{5}{64}M_3m^6\frac{(1+p)^3}{(1+p')^3}[30\cos(\tau+w-w')+35\cos 3(\tau+w-w')\\
&\qquad\qquad+63\cos 5(\tau+w-w')]+\cdots\Big\},\\
&(1+p)\frac{d^2w}{d\tau^2}+2\frac{dp}{d\tau}\left(1+m+\frac{dw}{d\tau}\right)=-\frac{m^2\eta}{2}\frac{(1+p)}{(1+p')^3}\Big\{3\sin 2(\tau+w-w')\\
&\quad+\tfrac{3}{4}M_1m^2\frac{1+p}{1+p'}[\sin(\tau+w-w')+5\sin 3(\tau+w-w')]\\
&\quad+\tfrac{5}{8}M_2m^4\frac{(1+p)^2}{(1+p')^2}[2\sin 2(\tau+w-w')+7\sin 4(\tau+w-w')]\\
&\quad+\tfrac{15}{64}M_3m^6\frac{(1+p)^3}{(1+p')^3}[2\sin(\tau+w-w')+7\sin 3(\tau+w-w')\\
&\qquad\qquad+21\sin 5(\tau+w-w')]+\cdots\Big\},
\end{aligned}\qquad (9)$$

$$\frac{d^2p'}{d\tau^2} - (1+p')\left(m+\frac{dw'}{d\tau}\right)^2 + \frac{m^2}{(1+p')^2} = -\frac{EMN_2}{2(E+M)^2}m^6\frac{(1+p)^2}{(1+p')^4} \times \left\{[1+3\cos 2(\tau+w-w')]+\cdots\right\},$$

$$(1+p')\frac{d^2w'}{d\tau^2} + 2\frac{dp'}{d\tau}\left(m+\frac{dw'}{d\tau}\right) = \frac{EMN_2}{2(E+M)^2}m^6\frac{(1+p)^2}{(1+p')^4} \times \left\{3\sin 2(\tau+w-w')+\cdots\right\}.$$

These equations are general for the determination of the motion of three mutually attracting spheres except for the condition that $a(1+p)$ shall be less than $a'(1+p')$ throughout the motion.

3. Existence of the periodic solutions.

When $m=0$ equations (9) admit the periodic solution $p=w=p'=w'=0$. It will now be proved that for $m \neq 0$, but sufficiently small, the equations admit a periodic solution expansible as power series in m and vanishing with m. It is the analytic continuation of the solution $p=w=p'=w'=m=0$ with respect to m as the parameter. Since τ enters explicitly in (9) under the sine and cosine functions in integral multiples, it follows that the period of the solution must be a multiple of 2π.

It can easily be shown from the way in which p, w, p', w', and τ enter in (9) that if we impose the conditions $dp/d\tau = w = dp'/d\tau = w = 0$ at $\tau = j\pi$, then p and p' are even functions of $\tau - j\pi$, and w and w' are odd functions of $\tau - j\pi$. This means geometrically that at $\tau = j\pi$ the bodies are in a line and are crossing it perpendicularly, and that the orbits of M and S are respectively symmetrical with respect to axes coinciding with this line at $\tau = j\pi$ and rotating around E and the center of gravity of E and M at any constant rate. Orbits of this type, whether they are periodic or not, will be called *symmetrical orbits*.

The existence of symmetrical periodic orbits will be established first. Suppose that, at $\tau = 0$,

$$r = a(1+p) = a(1+\alpha)(1-e), \quad r' = a'(1+p') = a(1+\alpha')(1-e'),$$

$$(10) \qquad \frac{dr}{d\tau} = a\frac{dp}{d\tau} = 0, \qquad \frac{dr'}{d\tau} = a'\frac{dp'}{d\tau} = 0,$$

$$w = 0, \qquad w' = 0,$$

$$1+m+\frac{dw}{d\tau} = \frac{(1+m)\sqrt{1+e}}{(1+\alpha)^{\frac{3}{2}}(1-e)^{\frac{3}{2}}}, \qquad m+\frac{dw'}{d\tau} = \frac{m\sqrt{1+e'}}{(1+\alpha')^{\frac{3}{2}}(1-e')^{\frac{3}{2}}},$$

If the right members of equations (9) are neglected, and if we put $r = a(1+p)$, $(1+m)\tau + w = u$, $r' = a'(1+p')$, $m\tau + w' = u'$, then the left members reduce

to two sets of equations for the two body problem. It is found that, with the initial conditions (10), $a\alpha$, and $a'\alpha'$ are the increments to the semi-axes of the orbits of M and S respectively, and that e and e' are the eccentricities of their orbits. Because of the well-known properties of the solutions in terms of these parameters, the properties of the general solutions so far as they do not depend upon the right members of (9) can easily be written down. For this reason these particular parameters were selected.

Equations (9) are to be integrated as power series in α, e, α', e' and m, vanishing with these parameters. It follows from their form and POINCARÉ's extension* of CAUCHY's theorem that, when an arbitrary interval for τ has been chosen in advance, these parameters can be assigned such small moduli that the series will converge uniformly as τ takes all values in the interval. We shall choose as the interval $0 \cdots 2k\pi$ (k an integer) and the solution may be written

$$
\begin{aligned}
p &= \mathfrak{p}_1(\alpha, e, \alpha', e', m; \tau), \\
\frac{dp}{d\tau} &= \mathfrak{p}_2(\alpha, e, \alpha', e', m; \tau), \\
w &= \mathfrak{p}_3(\alpha, e, \alpha', e', m; \tau), \\
\frac{dw}{d\tau} &= \mathfrak{p}_4(\alpha, e, \alpha', e', m; \tau), \\
p' &= \mathfrak{p}_5(\alpha, e, \alpha', e', m; \tau), \\
&\cdots\cdots\cdots \\
\frac{dw'}{d\tau} &= \mathfrak{p}_8(\alpha, e, \alpha', e', m; \tau),
\end{aligned} \tag{11}
$$

where $\mathfrak{p}_1, \cdots, \mathfrak{p}_8$ are power series in α, e, α', e', and m. The independent variable τ enters in the coefficients.

It follows from the symmetry of the orbits under the initial conditions (10) that if at $\tau = k\pi$ the three bodies are again in a line and are crossing the line perpendicularly, then the orbits necessarily reënter at $\tau = 2k\pi$, and conversely. Hence the necessary and sufficient conditions for a symmetrical periodic solution of period $2k\pi$ are

$$
\mathfrak{p}_i(\alpha, e, \alpha', e', m; k\pi) = 0 \qquad (i = 2, 3, 6, 7). \tag{12}
$$

Since $\mathfrak{p}_2 = \mathfrak{p}_3 = \mathfrak{p}_6 = \mathfrak{p}_7 = 0$ at $\tau = 0$, it follows that equations (12) are identically satisfied by $\alpha = e = \alpha' = e' = m = 0$. It will be shown that they can be solved for α, e, α' and e' as power series in m, vanishing with m, and converging for sufficiently small values of m. When the results of these solutions

*Acta Mathematica, vol. 13 (1890), p. 16; *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, p. 58.

are substituted in (11), the coördinates will be expressed as power series in m alone, and they will converge for sufficiently small values of m. The conditions for periodicity will have been satisfied, and therefore for m sufficiently small, but not zero, the series will converge uniformly for all values of τ. *Since the series converge for a continuous range of values of m, the coefficient of each power will be periodic.*

The first two equations of (9) do not carry p' and w' except in their right members; therefore $\mathfrak{p}_1, \cdots, \mathfrak{p}_4$ *do not contain α' and e' except in terms which are multiplied by m^2. Likewise $\mathfrak{p}_5, \cdots, \mathfrak{p}_8$ do not involve α and e except in terms which are multiplied by m^6.* All the terms in $\mathfrak{p}_1, \cdots, \mathfrak{p}_4$ which do not contain m^2 as a factor can be obtained from the left members of the first two equations of (9) set equal to zero. All the terms in $\mathfrak{p}_5, \cdots, \mathfrak{p}_8$ which do not contain m^6 as a factor can be obtained from the left members of the third and fourth equations of (9) set equal to zero. The left members of (9) set equal to zero constitute two pairs of equations of the two body problem, and making use of this fact and the initial conditions (10), we may write the solution (11):

$$
\begin{aligned}
p &= \mathfrak{p}_1 = -e\left\{\cos \nu\tau + \frac{e}{2}(\cos 2\nu\tau - 1) + \cdots\right\} + m^2\mathfrak{p}_1'(\alpha, e, \alpha', e', m; \tau),\\
\frac{dp}{d\tau} &= \mathfrak{p}_2 = e\nu\left\{\sin \nu\tau + e \sin 2\nu\tau + \cdots\right\} + m^2\mathfrak{p}_2'(\alpha, e, \alpha', e', m; \tau),\\
w &= \mathfrak{p}_3 = \left\{-(1+m)\tau + \nu\tau + 2e \sin \nu\tau + \cdots\right\} + m^2\mathfrak{p}_3'(\alpha, e, \alpha', e', m; \tau)\\
\frac{dw}{d\tau} &= \mathfrak{p}_4 = \left\{-(1+m) + \nu + 2e\nu \cos \nu\tau + \cdots\right\} + m^2\mathfrak{p}_4'(\alpha, e, \alpha', e', m; \tau),\\
p' &= \mathfrak{p}_5 = -e'\left\{\cos \nu'\tau + \frac{e'}{2}(\cos 2\nu'\tau - 1) + \cdots\right\} + m^6\mathfrak{p}_5'(\alpha, e, \alpha', e', m; \tau),\\
\frac{dp'}{d\tau} &= \mathfrak{p}_6 = e'\nu'\left\{\sin \nu'\tau + e' \sin 2\nu'\tau + \cdots\right\} + m^6\mathfrak{p}_6'(\alpha, e, \alpha', e', m; \tau),\\
w' &= \mathfrak{p}_7 = \left\{-m\tau + \nu'\tau + 2e' \sin \nu'\tau + \cdots\right\} + m^6\mathfrak{p}_7'(\alpha, e, \alpha', e', m; \tau),\\
\frac{dw'}{d\tau} &= \mathfrak{p}_8 = \left\{-m + \nu' + 2e'\nu' \cos \nu'\tau + \cdots\right\} + m^6\mathfrak{p}_8'(\alpha, e, \alpha', e', m; \tau),
\end{aligned}
\tag{13}
$$

where

$$
\nu = \frac{1+m}{(1+\alpha)^{\frac{3}{2}}}, \qquad \nu' = \frac{m}{(1+\alpha')^{\frac{3}{2}}}. \tag{14}
$$

The terms contained in $\mathfrak{p}_1', \cdots, \mathfrak{p}_8'$ depend upon the right members of the differential equations.

The conditions for periodic solutions (12) become

$$
\begin{aligned}
&\mathfrak{p}_2(k\pi) = ev\{\sin \nu k\pi + e \sin 2\nu k\pi + \cdots\} + m^2\mathfrak{p}_2'(\alpha, e, \alpha', e', m;\ k\pi) = 0,\\
&\mathfrak{p}_3(k\pi) = \{-(1+m)k\pi + \nu k\pi + 2e \sin \nu k\pi + \cdots\}\\
&\qquad\qquad + m^2\mathfrak{p}_3'(\alpha, e, \alpha', e', m;\ k\pi) = 0,\\
&\mathfrak{p}_6(k\pi) = e'\nu\{\sin \nu' k\pi + e' \sin 2\nu' k\pi + \cdots\} + m^6\mathfrak{p}_6'(\alpha, e, \alpha', e', m;\ k\pi) = 0,\\
&\mathfrak{p}_7(k\pi) = \{-mk\pi + \nu' k\pi + 2e' \sin \nu' k\pi + \cdots\}\\
&\qquad\qquad + m^6\mathfrak{p}_7'(\alpha, e, \alpha', e', m;\ k\pi) = 0.
\end{aligned}
\tag{15}
$$

All the unwritten terms in the $\{\ \}$ involve sines of multiples of $\nu k\pi$ or $\nu' k\pi$; those in the first two $\{\ \}$ carry e^2 as a factor, and in the last two, e'^2.

Substituting the expression for ν' as defined in (14) in the last two equations of (15), we find that $\mathfrak{p}_6(k\pi)$ is divisible by m^2 and that $\mathfrak{p}_7(k\pi)$ is divisible by m. After division by these factors there is no term independent of α', e', and m, and the determinant of the linear terms in α' and e' is

$$
\Delta \equiv \begin{vmatrix} 0, & k\pi \\ -\frac{3}{2}k\pi, & 2k\pi \end{vmatrix},
$$

which is distinct from zero. An exception can occur only if mk is an integer, when the determinant does vanish. But since k may be put equal to unity this requires that m shall be an integer, a case which arises for no satellite and for which the series derived are almost certainly always divergent. Hence it will be supposed that m is not an integer. Since Δ is not zero, $\mathfrak{p}_6(k\pi)/m^2 = 0$ and $\mathfrak{p}_7(k\pi)/m^2 = 0$ may be solved uniquely for α' and e' as power series in α, e, and m. These equations are satisfied by $\alpha' = e' = m = 0$ whatever α and e may be, and therefore the solutions contain m as a factor, and may be written

$$
\alpha' = mq_1(\alpha, e, m), \qquad e' = mq_2(\alpha, e, m).
$$

When these expressions for α' and e' are substituted in the first two equations of (15) the latter become

$$
\begin{aligned}
&\mathfrak{p}_2(k\pi) = ev\{\sin \nu k\pi + e \sin 2\nu k\pi + \cdots\} + m^2\mathfrak{p}_2''(\alpha, e, m;\ k\pi) = 0,\\
&\mathfrak{p}_3(k\pi) = \{-(1+m)k\pi + \nu k\pi + 2e \sin \nu k\pi + \cdots\} + m^2\mathfrak{p}_3''(\alpha, e, m;\ k\pi) = 0.
\end{aligned}
\tag{16}
$$

Before solving equations (16) we shall find the degree of the lowest terms in m alone which they contain. The terms which are independent of α, e, α' and e' are found by integrating equations (9) in series of the form

$$
p = \sum_{i=1}^{\infty} p_i m^i, \qquad w = \sum_{i=1}^{\infty} w_i m^i, \qquad p' = \sum_{i=1}^{\infty} p_i' m^i, \qquad w' = \sum_{i=1}^{\infty} w_i' m^i.
$$

Substituting these expressions in (9), equating coefficients of corresponding powers of m, integrating the resulting equations, and imposing the initial conditions (10), we find

$$
\begin{aligned}
p_1 &= w_1 = 0, \\
p_2 &= -\eta + 2\eta \cos\tau - \eta \cos 2\tau, \\
w_2 &= \tfrac{5}{4}\eta\tau - 4\eta \sin\tau + \tfrac{11}{8}\eta \sin 2\tau, \\
p_3 &= \tfrac{1}{2}\eta + \tfrac{2}{3}\eta \cos\tau - \tfrac{7}{6}\eta \cos 2\tau - 2\eta\tau \sin\tau, \\
w_3 &= \eta\tau - \tfrac{4}{3}\eta \sin\tau + \tfrac{13}{6}\eta \sin 2\tau - 4\eta\tau \cos\tau.
\end{aligned}
$$

These expressions give, at $\tau = k\pi$,

$$
\frac{dp_1}{d\tau} = w_1 = \frac{dp_2}{d\tau} = 0, \qquad w_2 = \tfrac{5}{4}\eta k\pi,
$$

$$
\frac{dp_3}{d\tau} = -(-1)^k 2\eta k\pi, \qquad w_3 = \eta k\pi - (-1)^k 4\eta k\pi.
$$

Therefore the lowest power of m contained in the first equation of (16) is the third, and in the second equation, the second.

Substitute in (16) the expression for ν given in (14) and consider the solution of the second equation. Every term contains either m or α as a factor. The lowest degree in α alone is the first, and in m alone, the second. Therefore the equation can be solved for α as a power series in m and e, containing m^2 as a factor. Thus

$$
\alpha = m^2 q_3(e, m). \tag{17}
$$

Let (17) be substituted in the first equation of (16). After the substitution it is found that a single factor m can be divided out leaving a linear term in e and a second degree term in m as the lowest. Consequently the equation can be solved for e as a power series in m, containing m^2 as a factor. Therefore we have

$$
e = m^2 q_4(m). \tag{18}
$$

By substituting back through the several steps of the solution, we obtain

$$
\begin{aligned}
\alpha &= m^2 q_1'(m), \qquad e = m^2 q_2'(m), \\
\alpha' &= m^2 q_3'(m), \qquad e' = m^2 q_4'(m),
\end{aligned} \tag{19}
$$

where $q_1', \cdots, q_4'$ are power series in m. The first two contain terms independent of m; the question for the last two will be left open until the practical construction of the periodic series is considered, though it might be answered at this place.

When the expressions (19) are substituted in equations (13) the periodic solution is obtained. *The series for p and w start with terms of the second degree in m.* When k has been chosen the whole process is unique. Therefore for a given value of m there is but one direct orbit of the type under consideration which reënters after k synodic revolutions. This holds when $k = 1$. But all the reëntrant orbits after k revolutions include those which reënter after one revolution; therefore, since the result is unique for every k, *all of the orbits of the type treated reënter after a single synodic revolution.*

If the orbit of M is retrograde, n is negative and m has the definition

$$m = \frac{-n'}{n+n'}.$$

With this definition of m the discussion of the existence of symmetrical periodic orbits proceeds precisely as before, and it is found that they exist. Consequently, *there is but one direct and but one retrograde orbit of the type in question with a given sidereal period.* For the retrograde orbit the modulus of m is smaller than for the direct, and the deviations from a circular orbit are in general less.

Suppose the condition is no longer imposed that the orbits shall be symmetrical, and consider the question of the existence of periodic solutions. We may suppose that at $\tau = 0$

$$p = \alpha, \quad w = 0, \quad \frac{dp}{d\tau} = \beta, \quad \frac{dw}{d\tau} = \gamma,$$

$$p' = \alpha', \quad w' = 0, \quad \frac{dp'}{d\tau} = \beta', \quad \frac{dw'}{d\tau} = \gamma'.$$

That is, at the origin of time E, M and S are in a line, but the initial conditions are not otherwise defined. The conditions $w = w' = 0$ are no limitations upon the generality of the problem for they amount simply to a particular choice of the origin of time. The solutions may be expressed as power series in $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ and m whose moduli may be taken so small that the convergence holds for τ in the whole interval $0 \cdots 2\pi$. The conditions for periodic solutions with period 2π are

$$\begin{aligned}
&p(\alpha, \beta, \gamma, \alpha', \beta', \gamma';\ 2\pi) = p(\alpha, \beta, \gamma, \alpha', \beta', \gamma';\ 0) = \alpha, \\
&\frac{dp}{d\tau}(\alpha, \beta, \gamma, \alpha', \beta', \gamma';\ 2\pi) = \frac{dp}{d\tau}(\alpha, \beta, \gamma, \alpha', \beta', \gamma';\ 0) = \beta, \qquad (20) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{dw'}{d\tau}(\alpha, \beta, \gamma, \alpha', \beta', \gamma';\ 2\pi) = \frac{dw'}{d\tau}(\alpha, \beta, \gamma, \alpha', \beta', \gamma';\ 0) = \gamma'.
\end{aligned}$$

Equations (9) admit two uniform integrals, that of vis viva and that of areas. They may be written

$$F_1\left(p, \frac{dp}{d\tau}, w, \frac{dw}{d\tau}, p', \frac{dp'}{d\tau}, w', \frac{dw'}{d\tau}\right) = F_1(\alpha, \beta, \gamma, \alpha', \beta', \gamma', m),$$

$$F_2\left(p, \frac{dp}{d\tau}, w, \frac{dw}{d\tau}, p', \frac{dp'}{d\tau}, w', \frac{dw'}{d\tau}\right) = F_2(\alpha, \beta, \gamma, \alpha', \beta', \gamma', m).$$

It follows from these relations that when six of equations (20) are satisfied the remaining two are reduced to identities.* Consequently two of the equations may be suppressed and the remaining six solved for $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ in terms of m. These solutions furnish the periodic orbits.

In order to determine which equations may be most conveniently suppressed because of the existence of the integrals F_1 and F_2 some details are necessary. Likewise in solving the remaining six equations more suitable parameters must be employed. However, the details will all be omitted. It is known that the equations admit one solution for they include as a special case the symmetrical orbits. The detailed discussion shows that they admit but one solution. Therefore *the periodic orbits of the type in question are all symmetrical.*

4. Practical construction of the periodic solutions.

The demonstrations of the existence of the periodic solutions carry with them possible, but very laborious, methods of constructing the series. A direct and convenient method of computing the coefficients will now be developed.

It has been proved that the periodic solutions are expressible in the form

$$(21) \qquad p = \sum_{i=2}^{\infty} p_i m^i, \qquad w = \sum_{i=2}^{\infty} w_i m^i,$$

$$p' = \sum_{i=2}^{\infty} p'_i m', \qquad w' = \sum_{i=2}^{\infty} w'_i m^i.$$

The derivatives are expressible in the same form, and all the series converge for all values of m whose moduli are sufficiently small. It has been shown that the coefficients of (21) can be uniquely determined so that they shall fulfil the three following conditions:

(*a*) The series satisfy the differential equation (9); and, because these conditions are fulfilled uniformly over a finite interval in m, *they satisfy the differential equations identically in* m.

(*b*) The series are periodic; and, because the periodicity holds for a continuous range of values of m, *each* p_i, w_i, p'_i, w'_i *is separately periodic.*

* Compare *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, p. 88.

(*c*) Since the orbits are symmetrical the origin of time may be chosen so that $dp/d\tau = w = dp'/d\tau = w' = 0$ at $\tau = 0$. Since this is true for all values of m whose moduli are sufficiently small, *each* $dp_i/d\tau$, w_i, $dp_i'/d\tau$ *and* w_i' *separately vanishes at* $\tau = 0$.

These conditions will be referred to simply as (*a*), (*b*) and (*c*).

Equations (21) are to be substituted in (9) and the results rearranged as power series in m. It is most convenient to make these preliminary expansions as far as they will be required before the integration is taken up. In the induction which follows we shall need the terms in p and w to the fourth power, and in p' and w' to the sixth. Up to this order we have from (9) and (21)

$$+(1+p)\left(1+m+\frac{dw}{d\tau}\right)^2 = [1]+[2]m+\left[1+2\frac{dw_2}{d\tau}+p_2\right]m^2$$
$$+\left[2\left(\frac{dw_3}{d\tau}+\frac{dw_2}{d\tau}+p_2\right)+p_3\right]m^3$$
$$+\left[2\left(\frac{dw_4}{d\tau}+\frac{dw_3}{d\tau}+p_2\frac{dw_2}{d\tau}+p_3\right)+\left(\frac{dw_2}{d\tau}\right)^2+p_2+p_4\right]m^4+\cdots,$$

$$-\frac{(1+m)^2}{(1+p)^2} = [-1]+[-2]m+[-1+2p_2]m^2+[2p_3+4p_2]m^3$$
$$+[2p_4-3p_2^2+4p_3+2p_2]m^4+\cdots,$$

$$+\frac{m^2\eta}{2}\frac{(1+p)}{(1+p')^3}[1+3\cos 2(\tau+w-w')] = \frac{m^2\eta}{2}\{[1+3\cos 2\tau]$$
$$+[p_2(1+3\cos 2\tau)-6w_2\sin 2\tau]m^2+\cdots\},$$

$$(22)\quad +\tfrac{3}{8}M_1m^4\eta\frac{(1+p)^2}{(1+p')^4}[3\cos(\tau+w-w')+5\cos 3(\tau+w-w')]$$
$$=\tfrac{3}{8}M_1m^4\eta\{[3\cos\tau+5\cos 3\tau]+\cdots\},$$

$$-(1+p)\frac{d^2w}{d\tau^2} = -\left[\frac{d^2w_2}{d\tau^2}\right]m^2-\left[\frac{d^2w_3}{d\tau^2}\right]m^3-\left[\frac{d^2w_4}{d\tau^2}+p_2\frac{d^2w_2}{d\tau^2}\right]m^4+\cdots,$$

$$-2\frac{dp}{d\tau}\left(1+m+\frac{dw}{d\tau}\right) = -\left[2\frac{dp_2}{d\tau}\right]m^2-\left[2\frac{dp_3}{d\tau}+2\frac{dp_2}{d\tau}\right]m^3$$
$$-\left[2\frac{dp_4}{d\tau}+2\frac{dp_3}{d\tau}+2\frac{dp_2}{d\tau}\frac{dw_2}{d\tau}\right]m^4+\cdots,$$

$$-\tfrac{3}{2}m^2\eta\frac{(1+p)}{(1+p')^3}\sin 2(\tau+w-w') = -\tfrac{3}{2}m^2\eta\{[\sin 2\tau]$$
$$+[p_2\sin 2\tau+2w_2\cos 2\tau]m^2+\cdots\},$$

$$-\tfrac{3}{8}M_1m^4\eta\frac{(1+p)^2}{(1+p')^4}[\sin(\tau+w-w')+5\sin 3(\tau+w-w')]$$
$$=-\tfrac{3}{8}M_1\eta m^4\{[\sin\tau+5\sin 3\tau]+\cdots\},$$

and corresponding equations from the last two of (9) which need not be written down. It is a simple matter to write out the expansions as far as is desired. In the computations given in this paper the series were carried out so as to include all terms in m^{10}. It will be shown that after these series are written out there is no need of further reference to the differential equations.

It has been proved that there are no terms of the first degree in m. This result will be accepted though it would have come out in the construction if it were not already known. From (22) and condition (a) it follows that the equations which define the coefficients of m^2 in the solution are

$$\frac{d^2p_2}{d\tau^2} - 3p_2 - 2\frac{dw_2}{d\tau} = \tfrac{1}{2}\eta(1 + 3\cos 2\tau), \qquad \frac{d^2p_2'}{d\tau^2} = 0,$$

$$\frac{d^2w_2}{d\tau^2} + 2\frac{dp_2}{d\tau} = -\tfrac{3}{2}\eta \sin 2\tau, \qquad \frac{d^2w_2'}{d\tau^2} = 0.$$

The general solutions of these equations are

$$\begin{aligned} p_2 &= \tfrac{1}{2}\eta + 2c_1^{(2)} + c_2^{(2)}\cos\tau + c_3^{(2)}\sin\tau - \eta\cos 2\tau, \\ w_2 &= c_4^{(2)} - (3c_1^{(2)} + \eta)\tau - 2c_2^{(2)}\sin\tau + 2c_3^{(2)}\cos\tau + \tfrac{11}{8}\eta\sin 2\tau, \\ p_2' &= c_1'^{(2)}\tau + c_2'^{(2)}, \\ w_2' &= c_3'^{(2)}\tau + c_4'^{(2)}, \end{aligned} \tag{23}$$

where $c_1^{(2)}$, $c_2^{(2)}$, $c_3^{(2)}$, $c_4^{(2)}$, $c_1'^{(2)}$, $c_2'^{(2)}$, $c_3'^{(2)}$ and $c_4'^{(2)}$ are the constants of integration which are so far arbitrary.

It follows from condition (b) that

$$3c_1^{(2)} + \eta = 0, \qquad c_1'^{(2)} = c_3'^{(2)} = 0;$$

and from condition (c) that

$$c_3^{(2)} = c_4^{(2)} = c_4'^{(2)} = 0.$$

Therefore equations (23) become

$$\begin{aligned} p_2 &= -\tfrac{1}{6}\eta + c_2^{(2)}\cos\tau - \eta\cos 2\tau, & p_2' &= c_2'^{(2)}, \\ w_2 &= -2c_2^{(2)}\sin\tau + \tfrac{11}{8}\eta\sin 2\tau, & w_2' &= 0, \end{aligned} \tag{24}$$

where $c_2^{(2)}$ and $c_2'^{(2)}$ are so far arbitrary.

The equations which determine the coefficients of m^3 are found from (22) to be

$$\frac{d^2p_3}{d\tau^2} - 3p_3 - 2\frac{dw_3}{d\tau} = 2\frac{dw_2}{d\tau} + 6p_2 = -\eta + 2c_2^{(2)}\cos\tau - \tfrac{1}{2}\eta\cos 2\tau,$$

$$\frac{d^2w_3}{d\tau^2} + 2\frac{dp_3}{d\tau} = -2\frac{dp_2}{d\tau} = 2c_2^{(2)}\sin\tau - 4\eta\sin 2\tau,$$

$$\frac{d^2p_3'}{d\tau^2} = 0,$$

$$\frac{d^2w_3'}{d\tau^2} = 0.$$

The general solutions of these equations are

$$p_3 = 2c_1^{(3)} - \eta + c_2^{(3)}\cos\tau + c_3^{(3)}\sin\tau - \tfrac{7}{6}\eta\cos 2\tau - c_2^{(2)}\tau\sin\tau,$$

$$w_3 = c_4^{(3)} + (-3c_1^{(3)} + 2\eta)\tau - 2c_2^{(3)}\sin\tau + 2c_3^{(3)}\cos\tau + \tfrac{13}{6}\eta\sin 2\tau - 2c_2^{(2)}\tau\cos\tau,$$

$$p_3' = c_1'^{(3)}\tau + c_2'^{(3)},$$

$$w_3' = c_3'^{(3)}\tau + c_4'^{(3)},$$

where $c_1^{(3)}, \cdots, c_4'^{(3)}$ are so far arbitrary.

From conditions (b) and (c) we have

$$c_2^{(2)} = c_3^{(3)} = c_4^{(3)} = c_1'^{(3)} = c_3'^{(3)} = c_4'^{(3)} = 0,$$
$$-3c_1^{(3)} + 2\eta = 0,$$

while $c_2^{(3)}$ and $c_2'^{(3)}$ so far arbitrary.

Hence we have

$$p_2 = -\tfrac{1}{6}\eta - \eta\cos 2\tau, \qquad p_2' = c_2'^{(2)},$$
$$w_2 = \tfrac{11}{8}\eta\sin 2\tau, \qquad w_2' = 0,$$
$$p_3 = \tfrac{1}{3}\eta + c_2^{(3)}\cos\tau - \tfrac{7}{6}\eta\cos 2\tau, \qquad p_3' = c_2'^{(3)},$$
$$w_3 = -2c_2^{(3)}\sin\tau + \tfrac{13}{6}\eta\sin 2\tau, \qquad w_3' = 0;$$

p_2 and w_2 are now completely determined.

It is necessary to go one step further in order to make the induction to the general formulas. The equations which determine the terms of the fourth order are, from (22),

$$\frac{d^2p_4}{d\tau^2} - 3p_4 - 2\frac{dw_4}{d\tau} = 2\frac{dw_3}{d\tau} + 2p_2\frac{dw_2}{d\tau} - 3p_2^2 + 6p_3 + \left(\frac{dw_2}{d\tau}\right)^2 + 3p_2$$
$$+\tfrac{1}{2}\eta p_2 + \tfrac{3}{2}\eta p_2\cos 2\tau - 3\eta w_2\sin 2\tau + \tfrac{9}{8}M_1\eta\cos\tau + \tfrac{15}{8}M_1\eta\cos 3\tau$$
$$= [-\tfrac{33}{96}\eta^2 + \tfrac{3}{2}\eta] + [2c_2^{(3)} + \tfrac{9}{8}M_1\eta]\cos\tau - \tfrac{4}{3}\eta[1 + 2\eta]\cos 2\tau$$
$$+ \tfrac{15}{8}M_1\eta\cos 3\tau + \tfrac{27}{32}\eta^2\cos 4\tau,$$

$$(26)\quad \frac{d^2w_4}{d\tau^2} + 2\frac{dp_4}{d\tau} = -p_2\frac{d^2w_2}{d\tau^2} - 2\frac{dp_3}{d\tau} - 2\frac{dp_2}{d\tau}\frac{dw_2}{d\tau} - \tfrac{3}{2}\eta p_2 \sin 2\tau$$
$$- 3\eta w_2 \cos 2\tau - \tfrac{3}{8}M_1\eta \sin \tau - \tfrac{15}{8}M_1\eta \sin 3\tau$$
$$= [2c_2^{(3)} - \tfrac{3}{8}M_1\eta] \sin \tau - \tfrac{3}{2}\eta(7+\eta) \sin 2\tau - \tfrac{15}{8}M_1\eta \sin 3\tau - \tfrac{15}{16}\tfrac{3}{2}\eta^2 \sin 4\tau,$$

$$\frac{d^2p_4'}{d\tau^2} = 3p_2' = 3c_2'^{(2)},$$

$$\frac{d^2w_4'}{d\tau^2} = 0.$$

Integrating the second equation once and substituting the expression for $dw_4/d\tau$ in the first, it is found that

$$(27)\quad \frac{d^2p_4}{d\tau^2} + p_4 = [2c_1^{(4)} - \tfrac{3}{9}\tfrac{3}{6}\tfrac{1}{}\eta^2 + \tfrac{3}{2}\eta] + [-2c_2^{(3)} + \tfrac{15}{8}M_1\eta] \cos \tau$$
$$+ \tfrac{3}{8}\eta(5 - 3\eta) \cos 2\tau + \tfrac{25}{8}M_1\eta \cos 3\tau + \tfrac{45}{8}\eta^2 \cos 4\tau.$$

In order that the solution of this equation may be periodic [condition (b)] the coefficient of $\cos \tau$ must be put equal to zero, or

$$(28)\qquad c_2^{(3)} = \tfrac{15}{16}M_1\eta.$$

Thus the undetermined coefficient which remained in p_3 and w_3 is uniquely determined by the condition that p_4 shall be periodic.

After $c_2^{(3)}$ has been determined by (28) the solution of (27) is periodic. When the condition has been imposed that $dp_4/d\tau$ shall equal zero at $\tau = 0$, the expression for p_4 will carry, in addition to $c_1^{(4)}$, an undetermined coefficient $c_2^{(4)}$ with $\cos \tau$. When this value of p_4 is substituted in the integral of the second equation of (26) it takes the form

$$\frac{dw_4}{d\tau} = [-3c_1^{(4)} + \tfrac{3}{4}\tfrac{3}{8}\tfrac{1}{}\eta^2 - 3\eta] + [\ ] \cos \tau + \cdots.$$

In order that w_4 may be periodic the constant part of the right member of this equation must vanish, which uniquely determines $c_1^{(4)}$. Then p_4 and w_4 carry a single undetermined constant which is determined in the next step.

Referring to the last two equations of (26) we find

$$p_4' = \tfrac{3}{2}c_2'^{(2)}t^2 + c_1'^{(4)}t + c_2'^{(4)}, \qquad w_4' = c_3'^{(4)}t + c_4'^{(4)}.$$

By conditions (b) and (c) we have

$$c_2'^{(2)} = c_1'^{(4)} = c_3'^{(4)} = c_4'^{(4)} = 0.$$

Therefore p_2', equation (24), is zero. Similarly p_3' can be shown to be zero. Consider the equations which define p_6' and w_6'. They are

$$\frac{d^2p_6'}{d\tau^2} = 3p_4' - \frac{EMN_2}{2(E+M)^2} - \frac{3EMN_2}{2(E+M)^2}\cos 2\tau,$$

$$\frac{d^2w_6'}{d\tau^2} = \frac{3EMN_2}{2(E+M)^2}\sin 2\tau,$$

where $p_4' = c_2'^{(4)}$. In order that p_6' may be periodic the constant part of the right member of the first of these equations must vanish. Therefore

$$p_4' = c_2'^{(4)} = \frac{EMN_2}{6(E+M)^2}. \tag{29}$$

Then it is found, after imposing condition (c), that

$$p_6' = c_1'^{(6)} + \frac{3EMN_2}{8(E+M)^2}\cos 2\tau, \qquad w_6' = -\frac{3EMN_2}{8(E+M)^2}\sin 2\tau. \tag{30}$$

We shall now develop general formulas which will obviate the necessity of integrating the equations at each step. Suppose $p_2, \cdots, p_{i-1}, w_2, \cdots, w_{i-1}$, $p_2', \cdots, p_{i-1}', w_2', \cdots, w_{i-1}'$ have been computed. Suppose

$$\begin{aligned} p_j &= \alpha_0^{(j)} + \alpha_1^{(j)}\cos\tau + \alpha_2^{(j)}\cos 2\tau + \cdots + \alpha_j^{(j)}\cos j\tau, \\ w_j &= \beta_1^{(j)}\sin\tau + \beta_2^{(j)}\sin 2\tau + \cdots + \beta_j^{(j)}\sin j\tau, \qquad (j=2,\cdots,i-1), \\ p_j' &= \alpha_0'^{(j)} + \alpha_1'^{(j)}\cos\tau + \alpha_2'^{(j)}\cos 2\tau + \cdots + \alpha_j'^{(j)}\cos(j-4)\tau, \\ w_j' &= \beta_1'^{(j)}\sin\tau + \beta_2'^{(j)}\sin 2\tau + \cdots + \beta_{j-4}'^{(j)}\sin(j-4)\tau. \end{aligned} \tag{31}$$

Suppose the coefficients are all uniquely determined except $\alpha_1^{(i-1)}$, $\beta_1^{(i-1)}$, which we shall suppose carry an undetermined constant linearly, and $\alpha_0'^{(i-2)}$ and $\alpha_0'^{(i-1)}$, which we shall suppose are so far arbitrary. The problem is to find the coefficients in the expressions for p_i, w_i, p_i' and w_i', and to determine $\alpha_1^{(i-1)}$, $\beta_1^{(i-1)}$, $\alpha_0'^{(i-2)}$ and $\alpha_0'^{(i-1)}$.

Consider the differential equations for the determination of p_i, w_i, p_i' and w_i'. The left members of the differential equations are of the same form for all values of i beyond 1, for example, as given in (26). The right members are polynomials in $p_1, \cdots, p_{i-1}, w_1, \cdots, w_{i-1}, p_1', \cdots, p_{i-1}', w_1', \cdots, w_{i-1}'$, and linear in $\cos j\tau$ and $\sin j\tau$. In any term which is independent of $\cos j\tau$ and $\sin j\tau$ the sum of the products of the exponents and subscripts cannot exceed i. In any term which involves $\cos j\tau$ or $\sin j\tau$ the sum of the products of the exponents and subscripts cannot exceed $i-j$. It follows from the way in which p, w, p', w'

and τ enter in the differential equations (9) that the right members of the first and third equations which determine p_i, w_i, p_i' and w_i' are even in the $dp_j/d\tau$, w_j, $dp_j'/d\tau$, w_j' and τ taken together, and that the second and fourth equations are odd in the same quantities. Hence it follows from the forms of equations (31) that when the values are substituted for the p_j, w_j, p_j' and w_j', the right members of the first and third equations will be sums of cosines of multiples of τ, and the right members of the second and fourth equations sums of sines of multiples of τ.

It is necessary to find how p_{i-1}, w_{i-1}, and p_{i-2}' enter in the right members of the differential equations which determine p_i, w_i, p_i', and w_i'. It is seen in the special case (26), and it can be shown in general from equations (9), that the first equation contains the terms $6p_{i-1}$ and $2\,dw_{i-1}/d\tau$, the second $-2\,dp_{i-1}/d\tau$, and the third $3p_{i-2}'$. These are the only terms that contain coefficients which are not completely determined. Hence the equations which determine p_i, w_i, p_i', and w_i' have the form

$$
\begin{aligned}
&\frac{d^2p_i}{d\tau^2} - 3p_i - 2\frac{dw_i}{d\tau} = A_0^{(i)} + (6\alpha_1^{(i-1)} + 2\beta_1^{(i-1)} + A_1^{(i)})\cos\tau \\
&\qquad\qquad + A_2^{(i)}\cos 2\tau + \cdots + A_i^{(i)}\cos i\tau, \\
&\frac{d^2w_i}{d\tau^2} + 2\frac{dp_i}{d\tau} = (2\alpha_1^{(i-1)} + B_1^{(i)})\sin\tau + B_2^{(i)}\sin 2\tau + \cdots + B_i^{(i)}\sin i\tau, \\
&\frac{d^2p_i'}{d\tau^2} = (3\alpha_0^{\prime(i-} + A_0^{\prime(i)}) + A_1^{\prime(i)}\cos\tau + \cdots + A_{i-4}^{\prime(i)}\cos(i-4)\tau, \\
&\frac{d^2w_i'}{d\tau^2} = B_1^{\prime(i)}\sin\tau + \cdots + B_{i-4}^{\prime(i)}\sin(i-4)\tau,
\end{aligned}
\tag{32}
$$

where the $A_j^{(i)}$, $B_j^{(i)}$, $A_j^{\prime(i)}$, and $B_i^{\prime(i)}$ are known numbers depending upon the coefficients of $p_2, \cdots, p_{i-1}, w_2, \cdots, w_{i-1}, p_2', \cdots, p_{i-1}', w_2', \cdots, w_{i-1}'$. The constants $\alpha_1^{(i-1)}$, $\beta_1^{(i-1)}$ and $\alpha_0^{\prime(i-2)}$ are as yet undetermined.

The solutions of (32) are to be periodic, and $dp_i/d\tau$, w_i, $dp_i'/d\tau$, and w_i' must vanish at $\tau = 0$. Therefore they must have the form

$$
\begin{aligned}
p_i &= \alpha_0^{(i)} + \alpha_1^{(i)}\cos\tau + \alpha_2^{(i)}\cos 2\tau + \cdots + \alpha_i^{(i)}\cos i\tau, \\
w_i &= \beta_1^{(i)}\sin\tau + \beta_2^{(i)}\sin 2\tau + \cdots + \beta_i^{(i)}\sin i\tau, \\
p_i' &= \alpha_0^{\prime(i)} + \alpha_1^{\prime(i)}\cos\tau + \alpha_2^{\prime(i)}\cos 2\tau + \cdots + \alpha_{i-4}^{\prime(i)}\cos(i-4)\tau, \\
w_i' &= \beta_1^{\prime(i)}\sin\tau + \beta_2^{\prime(i)}\sin 2\tau + \cdots + \beta_{i-4}^{\prime(i)}\sin(i-4)\tau.
\end{aligned}
\tag{33}
$$

The $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $\alpha_j^{\prime(i)}$, and $\beta_j^{\prime(i)}$ are to be determined in terms of the $A_j^{(i)}$, $B_j^{(i)}$, $A_j^{\prime(i)}$, $B_j^{\prime(i)}$, and the new constants of integration.

Consider first the third and fourth equations of (32). In order that p_i' may be periodic we must impose the condition

$$\alpha_0^{(i-2)} = -\tfrac{1}{2}A_0^{(i)}.$$

Then imposing conditions (b) and (c) we have

$$\alpha_0^{(i)} = \text{constant undetermined until computation of } p_{i+2},$$

$$\alpha_j^{(i)} = -\frac{1}{j^2}A_j^{(i)} \qquad (j=1, \cdots, i-4), \tag{34}$$

$$\beta_j^{(i)} = -\frac{1}{j^2}B_j^{(i)}.$$

Now consider the first two equations of (32). Integrating the second we have

$$\frac{dw_i}{d\tau} = c_1^{(i)} - 2p_i - (2\alpha_1^{(i-1)} + B_1^{(i)})\cos\tau - \tfrac{1}{2}B_2^{(i)}\cos 2\tau - \cdots - \frac{1}{i}B_i^{(i)}\cos i\tau. \tag{35}$$

Substituting this expression for $dw_i/d\tau$ in the first of (32), we find

$$\begin{aligned}\frac{d^2p_i}{d\tau^2} + p_i = {} & [A_0^{(i)} + 2c_1^{(i)}] + [A_1^{(i)} - 2B_1^{(i)} + 2\alpha_1^{(i-1)} + 2\beta_1^{(i-1)}]\cos\tau \\ & + \left[A_2^{(i)} - \frac{2}{2}B_2^{(i)}\right]\cos 2\tau + \cdots + \left[A_j^{(i)} - \frac{2}{j}B_j^{(i)}\right]\cos j\tau + \cdots \\ & + \left[A_i^{(i)} + \frac{2}{i}B_i^{(i)}\right]\cos i\tau.\end{aligned} \tag{36}$$

In order that the solution of this equation may be periodic [condition (b)] we must impose the condition

$$A_1^{(i)} - 2B_1^{(i)} + 2\alpha_1^{(i-1)} + 2\beta_1^{(i-1)} = 0. \tag{37}$$

Then the most general solution of (36) satisfying condition (c) [$dp_i/d\tau = 0$ at $\tau = 0$] is

$$\begin{aligned}p_i = {} & [A_0^{(i)} + 2c_1^{(i)}] + \alpha_1^{(i)}\cos\tau - \frac{[2A_2^{(i)} - 2B_2^{(i)}]}{2(2^2-1)}\cos 2\tau - \cdots \\ & - \frac{[jA_j^{(i)} - 2B_j^{(i)}]}{j(j^2-1)}\cos j\tau - \cdots - \frac{[iA_i^{(i)} - 2B_i^{(i)}]}{i(i^2-1)}\cos i\tau.\end{aligned} \tag{38}$$

Substituting this value of p_i in (35), we get

$$\begin{aligned}\frac{dw_i}{d\tau} = {} & -[2A_0^{(i)} + 3c_1^{(i)}] - [B_1^{(i)} + 2\alpha_1^{(i)} + 2\alpha_1^{(i-1)}]\cos\tau + \frac{[4A_2^{(i)} - 7B_2^{(i)}]}{2(2^2-1)}\cos 2\tau + \cdots \\ & + \frac{[2jA_j^{(i)} - (j^2+3)B_j^{(i)}]}{j(j^2-1)}\cos j\tau + \cdots + \frac{[2iA_i^{(i)} - (i^2+3)B_i^{(i)}]}{i(i^2-1)}\cos i\tau.\end{aligned}$$

In order that the solution of this equation may be periodic we must impose the condition

$$2A_0^{(i)} + 3c_1^{(i)} = 0, \tag{39}$$

which determines $c_1^{(i)}$ uniquely. Substituting in the first term in the right member of (38), we find $\alpha_0^{(i)} = A_0^{(i)} + 2c_1^{(i)} = -\frac{1}{3}A_0^{(i)}$. Then the most general solution of the equation for w_i, vanishing at $\tau = 0$, is

$$\begin{aligned} w_i = &-[B_1^{(i)} + 2\alpha_1^{(i)} + 2\alpha_1^{(i-1)}]\sin\tau + \frac{[4A_2^{(i)} - 7B_2^{(i)}]}{12}\sin 2\tau + \cdots \\ &+ \frac{[2jA_j^{(i)} - (j^2+3)B_j^{(i)}]}{j^2(j^2-1)}\sin j\tau + \cdots + \frac{[2iA_i^{(i)} - (i^2+3)B_i^{(i)}]}{i^2(i^2-1)}\sin i\tau. \end{aligned} \tag{40}$$

In the notation adopted in (33) the coefficient of $\sin\tau$ in equation (40) is $\beta_1^{(i)}$. Since the form is general we have

$$\beta_1^{(i-1)} = -[B_1^{(i-1)} + 2\alpha_1^{(i-1)} + 2\alpha_1^{(i-2)}]. \tag{41}$$

This equation used with (37) determines $\alpha_1^{(i-1)}$ and $\beta_1^{(i-1)}$ in terms of known quantities. Hence the general expressions for the coefficients of (33) are

$$\begin{aligned} \alpha_0^{(i)} &= -\tfrac{1}{3}A_0^{(i)} && (i=2,\cdots,\infty), \\ \alpha_1^{(i-1)} &= \tfrac{1}{2}[A_1^{(i)} - 2B_1^{(i)}] - [B_1^{(i-1)} + 2\alpha_1^{(i-2)}], \\ \beta_1^{(i-1)} &= -[A_1^{(i)} - 2B_1^{(i)}] + [B_1^{(i-1)} + 2\alpha_1^{(i-2)}], \\ \alpha_j^{(i)} &= -\frac{[jA_j^{(i)} - 2B_j^{(i)}]}{j(j^2-1)} && (j=2,\cdots,i), \\ \beta_j^{(i)} &= \frac{[2jA_j^{(i)} - (j^2+3)B_j^{(i)}]}{j^2(j^2-1)} && (j=2,\cdots,i), \\ \alpha_0'^{(i-2)} &= -\tfrac{1}{3}A_0'^{(i)} && (i=6,\cdots,\infty), \\ \alpha_j'^{(i)} &= -\frac{1}{j^2}A_j'^{(i)} && (j=1,\cdots,i-4), \\ \beta_j'^{(i)} &= -\frac{1}{j^2}B_j'^{(i)} \end{aligned} \tag{42}$$

The simplicity of the computation from these formulas and the expansions (22) will be illustrated by the computation of $\alpha_j^{(4)}$ and $\beta_j^{(4)}$.

TABLE I.

Term from (22).	Constant.	$\cos\tau$	$\cos 2\tau$	$\cos 3\tau$	$\cos 4\tau$
$+2\frac{dw_3}{d\tau}$	0	$-2\beta_1^{(3)}$	$+\frac{26}{3}\eta$	0	0
$+2p_3\frac{dw_2}{d\tau}$	$-\frac{11}{4}\eta^2$	0	$-\frac{11}{12}\eta^2$	0	$-\frac{11}{4}\eta^2$
$+6p_3$	$+2\eta$	$+6a_1^{(3)}$	-7η	0	0
$+\left(\frac{dw_3}{d\tau}\right)^2$	$+\frac{121}{32}\eta^2$	0	0	0	$+\frac{121}{32}\eta^2$
$+3p_2$	$-\frac{1}{2}\eta$	0	-3η	0	0
$-3p_2^2$	$-\frac{19}{12}\eta^2$	0	$-\eta^2$	0	$-\frac{3}{2}\eta^2$
$+\frac{1}{2}\eta p_2(1+3\cos 2\tau)$	$-\frac{5}{6}\eta^2$	0	$-\frac{3}{4}\eta^2$	0	$-\frac{3}{4}\eta^2$
$-3\eta w_2\sin 2\tau$	$-\frac{33}{16}\eta^2$	0	0	0	$+\frac{33}{16}\eta^2$
$+\frac{3}{8}\eta M_1(3\cos\tau+5\cos 3\tau)$	0	$+\frac{9}{8}\eta M_1$	0	$+\frac{15}{8}\eta M_1$	0
$A_j^{(4)}$	$+\frac{3}{2}\eta-\frac{3319}{96}\eta^2$	$+\frac{9}{8}\eta M_1$	$-\frac{4}{3}\eta-\frac{8}{3}\eta^2$	$+\frac{15}{8}\eta M_1$	$+\frac{27}{32}\eta^2$

TABLE II.

Term from (22).	$\sin\tau$	$\sin 2\tau$	$\sin 3\tau$	$\sin 4\tau$
$-p_2\frac{d^2w_2}{d\tau^2}$	0	$-\frac{11}{12}\eta^2$	0	$-\frac{11}{4}\eta^2$
$-2\frac{dp_3}{d\tau}$	$+2a_1^{(3)}$	$-\frac{14}{3}\eta$	0	0
$-2\frac{dp_3}{d\tau}\frac{dw_2}{d\tau}$	0	0	0	$-\frac{11}{2}\eta^2$
$-\frac{3}{2}\eta p_2\sin 2\tau$	0	$+\frac{1}{4}\eta^2$	0	$+\frac{3}{4}\eta^2$
$-3\eta w_2\cos 2\tau$	0	0	0	$-\frac{33}{16}\eta^2$
$-\frac{3}{8}\eta M_1(\sin\tau+5\sin 3\tau)$	$-\frac{3}{8}\eta M_1$	0	$-\frac{15}{8}\eta M_1$	0
$B_j^{(4)}$	$-\frac{3}{8}\eta M_1$	$-\frac{14}{3}\eta-\frac{2}{3}\eta^2$	$-\frac{15}{8}\eta M_1$	$-\frac{153}{16}\eta^2$

From these values of $A_j^{(4)}$ and $B_j^{(4)}$ ($B_j^{(3)}$ and $\alpha_1^{(2)}$ are zero) it is found by equations (42) that

$$
\begin{aligned}
&\alpha_0^{(4)} = -\frac{\eta}{2} + \frac{33}{28}\frac{1}{8}\eta^2,\\
&\alpha_1^{(3)} = +\frac{15}{16}\eta M_1, \qquad \beta_1^{(3)} = -\frac{15}{8}\eta M_1,\\
&\alpha_2^{(4)} = -\frac{10}{9}\eta + \frac{2}{3}\eta^2, \qquad \beta_2^{(4)} = +\frac{41}{18}\eta - \frac{1}{2}\eta^2,\\
&\alpha_3^{(4)} = -\frac{25}{64}\eta M_1, \qquad \beta_3^{(4)} = +\frac{15}{32}\eta M_1,\\
&\alpha_4^{(4)} = -\frac{3}{8}\eta^2, \qquad \beta_4^{(4)} = +\frac{201}{256}\eta^2.
\end{aligned}
\tag{43}
$$

As many succeeding $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ as may be desired may be computed in precisely

the same way. The operations are reduced to systematic routine, and the work consists almost wholly in multiplying fractions in constructing the tables, and in adding fractions in getting the $A_j^{(i)}$, $B_j^{(i)}$, $A_j'^{(i)}$ and $B_j'^{(i)}$.

The differential equations admit two integrals of which no use has been made. They may be used to derive new formulas for the computation of the $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $\alpha_j'^{(i)}$, and $\beta_j'^{(i)}$, or to verify the results obtained by (42). The principal applications occur when the mass of M is infinitesimal, or at the most so small that its effect on the motion of S around the center of gravity of E and M is insensible. Consequently, to abbreviate the discussion we shall assume that S moves in an undisturbed circle around the center of gravity of E and M. Then the integral of areas disappears and the vis viva integral becomes JACOBI's integral:

$$\left(\frac{dp}{d\tau}\right)^2+(1+p)^2\left(1+\frac{dw}{d\tau}\right)^2-m^2(1+p)^2-\frac{2(1+m)^2}{1+p}$$

$$-2m^2\eta\left(\frac{a'}{a}\right)^2\left\{\frac{1}{\sqrt{1-2\frac{a}{a'}(1+p)\cos(\tau+w)+\left(\frac{a}{a'}\right)(1+p)^2}}-\frac{a}{a}(1+p)\cos(\tau+w)\right\}=C, \tag{44}$$

where C is the Jacobian constant of integration.

In the periodic solution p and w are developed as power series in m, and hence we may write the integral for this solution in the form

$$F_0+F_1\left(p_1,\frac{dp_1}{d\tau},w_1,\frac{dw_1}{d\tau};\tau\right)m$$

$$+F_2\left(p_1,p_2,\frac{dp_1}{d\tau},\frac{dp_2}{d\tau},\cdots,\frac{dw_2}{d\tau};\tau\right)m^2+\cdots=C.$$

Since this integral converges for all values of m sufficiently small, it follows that each F_i is separately constant. In F_i the sum of the products of the exponents and subscripts of the factors of any term independent of $\cos j\tau$ cannot exceed i; and in any term involving $\cos j\tau$ the sum cannot exceed $i-j$. The function F is even in $dp_j/d\tau$, w_j and τ and therefore each F_i has the form

$$F_i=\gamma_0^{(i)}+\gamma_1^{(i)}\cos\tau+\cdots+\gamma_j^{(i)}\cos j\tau+\cdots+\gamma_i^{(i)}\cos i\tau. \tag{45}$$

Since every F_i is identically constant in τ it follows that

$$\gamma_j^{(i)}=0 \qquad (i=1,\cdots,\infty;\ j=1,\cdots,i). \tag{46}$$

The fulfillment of these relations, which are functions of $\alpha_j^{(2)},\cdots,\alpha_j^{(i)},\beta_j^{(2)},\cdots,\beta_j^{(i)}$ $(j=0,\cdots,i)$, serves as a complete check on the expansions and the computations.

The integral is applied by first expanding the separate terms of (44) as power series in m. Thus we have the expansions

$$\left(\frac{dp}{d\tau}\right)^2 = \left(\frac{dp_2}{d\tau}\right)^2 m^4 + 2\frac{dp_2}{d\tau}\frac{dp_3}{d\tau}m^5 + \cdots,$$

$$(1+p)^2\left(1+\frac{dw}{d\tau}\right)^2 = 1 + 2\left[p_2 + \frac{dw_2}{d\tau}\right]m^2 + 2\left[p_3 + \frac{dw_3}{d\tau}\right]m^3$$
$$+ \left[2p_4 + p_2^2 + 2\frac{dw_4}{d\tau} + \left(\frac{dw_2}{d\tau}\right)^2 + 4p_2\frac{dw_2}{d\tau}\right]m^4 + \cdots,$$

$$-m^2(1+p)^2 = -m^2 - 2p_2m^4 + \cdots,$$

$$-\frac{2(1+m)^2}{1+p} = -2 - 4m - 2[1-p_2]m^2 - 2[2p_2+p_3]m^3 \tag{47}$$
$$-2[p_4 - p_2^2 + 2p_3 + p_2]m^4 + \cdots,$$

$$-m^2\eta\left(\frac{a'}{a}\right)^2\left\{\left[1 - 2\frac{a}{a'}(1+p)\cos(\tau+w) + \left(\frac{a}{a'}\right)^2(1+p)^2\right]^{-\frac{1}{2}}\right.$$
$$\left. - \frac{a}{a'}(1+p)\cos(\tau+w)\right\} = -\tfrac{1}{2}\eta[1+3\cos 2\tau]m^2$$
$$+[-\eta p_2(1+3\cos 2\tau) + 3\eta w_2 \sin 2\tau - \tfrac{1}{4}M_1\eta(3\cos\tau + 5\cos 3\tau)]m^4 + \cdots.$$

It can be shown from these functions that in general F_i contains the terms $4p_i$, $2dw_i/d\tau$, $4p_{i-1}$, and terms with lower subscripts. Suppose the coefficients of $p_2, \cdots, p_{i-2}$, $w_2, \cdots, w_{i-2}$ are entirely known, and that the coefficients of p_{i-1}, and w_{i-1} are all known except $\alpha_1^{(i-1)}$ and $\beta_1^{(i-1)}$. Let in general the known part of $\gamma_j^{(i)}$ be $C_j^{(i)}$. Then the conditions (46) become

$$4\alpha_1^{(i)} + 2\beta_1^{(i)} + 4\alpha_1^{(i-1)} + C_1^{(i)} = 0 \qquad (i=2, \cdots, \infty),$$
$$4\alpha_j^{(i)} + 2j\beta_j^{(i)} + C_j^{(i)} = 0 \qquad (j=2, \cdots, i), \tag{48}$$

From equations (42) it is found that

$$4\alpha_1^{(i)} + 2\beta_1^{(i)} + 4\alpha_1^{(i-1)} + 2B_1^{(i)} = 0 \qquad (i=2, \cdots, \infty),$$
$$4\alpha_j^{(i)} + 2j\beta_j^{(i)} + \frac{2}{j}B_j^{(i)} = 0 \qquad (j=2, \cdots, i).$$

The comparison of these equations with (48) shows that

$$C_j^{(i)} - \frac{2}{j}B_j^{(i)} = 0 \qquad (i=2, \cdots, \infty;\ j=1, \cdots, i). \tag{49}$$

When the $B_j^{(i)}$ have been eliminated from equations (42) by means of these relations, the $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ are expressed in terms of the $A_j^{(i)}$ and $C_j^{(i)}$ by the equations

$$
\begin{aligned}
&\alpha_0^{(i)} = -\tfrac{1}{3}A_0^{(i)} && (i=2,\cdots,\infty),\\
&\alpha_1^{(i-1)} = \tfrac{1}{2}\left[A_1^{(i)} - C_1^{(i)}\right] - \left[\tfrac{1}{2}C_1^{(i-1)} + 2\alpha_1^{(i-2)}\right],\\
(50)\qquad &\beta_1^{(i-1)} = -\left[A_1^{(i)} - C_1^{(i)}\right] + \left[\tfrac{1}{2}C_1^{(i-1)} + 2\alpha_1^{(i-2)}\right],\\
&\alpha_j^{(i)} = -\frac{\left[A_j^{(i)} - C_j^{(i)}\right]}{j^2-1} && (j=2,\cdots,i),\\
&\beta_j^{(i)} = \frac{\left[4A_j^{(i)} - (j^2+3)C_j^{(i)}\right]}{2j(j^2-1)}.
\end{aligned}
$$

Therefore in case the orbit of S is a circle around the center of gravity of E and M the second differential equation, which involves the second derivative of w, may be entirely neglected in the determination of the coefficients of the periodic solution, and JACOBI's integral used in its place.* The more convenient process is to be employed. In every case there are more terms in the table for the $C_j^{(i)}$ than for the $B_j^{(i)}$, but a large proportion of the former are the same as those occurring in the table for the $A_j^{(i)}$ except for numerical factors. For this reason the $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ are more conveniently found from equations (50) than they are from equations (42), but then the advantage of having the integral as a check is lost. In the computations given in this paper the coefficients were obtained by (42), and the integral was used as a check.

The use of the integral will be illustrated in computing the coefficients of m^4.

TABLE III.

Term from (47).	Constant.	$\cos\tau$	$\cos 2\tau$	$\cos 3\tau$	$\cos 4\tau$
$+\left(\frac{dp_2}{d\tau}\right)^2$	$+2\eta^2$	0	0	0	$-2\eta^2$
$+4p_4$	$+4a_0^{(4)}$	$+4a_1^{(4)}$	$+4a_2^{(4)}$	$+4a_3^{(4)}$	$+4a_4^{(4)}$
$-p_2^2$	$-\frac{19}{36}\eta^2$	0	$-\frac{1}{3}\eta^2$	0	$-\frac{1}{2}\eta^2$
$+2\frac{dw_4}{d\tau}$	0	$+2\beta_1^{(4)}$	$+4\beta_2^{(4)}$	$+6\beta_3^{(4)}$	$+8\beta_4^{(4)}$
$+\left(\frac{dw_2}{d\tau}\right)^2$	$+\frac{121}{32}\eta^2$	0	0	0	$+\frac{121}{32}\eta^2$
$+4p_2\frac{dw_2}{d\tau}$	$-\frac{11}{2}\eta^2$	0	$-\frac{11}{6}\eta^2$	0	$-\frac{11}{2}\eta^2$
$-\eta p_2$	$+\frac{1}{6}\eta^2$	0	$+\eta^2$	0	0
$+4p_3$	$+\frac{4}{3}\eta$	$4a_1^{(3)}$	$-\frac{14}{3}\eta$	0	0
$-3\eta p_2\cos 2\tau$	$+\frac{3}{2}\eta^2$	0	$+\frac{1}{2}\eta^2$	0	$+\frac{3}{2}\eta^2$
$+3\eta w_2\sin 2\tau$	$+\frac{33}{16}\eta^2$	0	0	0	$-\frac{33}{16}\eta^2$
$-\frac{1}{4}M_1\eta(3\cos\tau+5\cos 3\tau)$	0	$-\frac{3}{4}M_1\eta$	0	$-\frac{5}{4}M_1\eta$	0
$C_j^{(4)}$	$+\frac{4}{3}\eta+\frac{2551}{72}\eta^2$	$-\frac{3}{4}M_1\eta$	$-\frac{14}{3}\eta-\frac{2}{3}\eta^2$	$-\frac{5}{4}M_1\eta$	$-\frac{1533}{32}\eta^2$

* HILL used the integral to eliminate the most troublesome expression arising in the differential equations in his method of treatment. Loc. cit., p. 132.

With these values of the $C_j^{(i)}$ and the $A_j^{(i)}$ of table I, equations (50) lead to the results of (43); or, when the values from (43) are substituted in table III, the relations (46) are satisfied.

Suppose the $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ are computed by (42) and that the integral is used as a check. It follows from (48) and (49) that at the ith step the check does not cover the $A_j^{(i)}$. That is, an error might exist in any $A_j^{(i)}$ without its being revealed by the check formula until the $(i+1)$st step. However, there is the check for the $A_j^{(i)}$ and $C_j^{(i)}$ that the sums of the coefficients in every line of their tables must equal the values of the functions in the first columns at $\tau = 0$, and the sums of the $A_j^{(i)}$ and $C_j^{(i)}$ (with respect to j) must equal the sums of their respective first columns at $\tau = 0$.

5. Applications to the lunar theory.

The differential equations and their solutions have been developed in such a form as to be immediately applicable to the Lunar Theory, provided we assume that its motion is periodic of the type under consideration. It is only necessary to find the order of magnitude of the parameters involved so as to know how far the developments must be carried to meet the needs of observers. Observations have given for the Lunar Theory the following approximate numerical relations:

$$n = 13.4n', \text{ whence } m = \frac{n'}{n - n'} = \frac{1}{12.4},$$

$$E = 80M,$$

$$(51)\quad S = 330{,}000E, \text{ whence } \eta = \frac{S}{S + E + M} = \frac{1}{1 + \frac{1}{300{,}000}},$$

$$\frac{a}{a'} = \tfrac{1}{400}, \text{ whence } M_1 = N_1 = \tfrac{3}{8},\ M_2 = N_2 = \tfrac{1}{7},\ M_3 = N_3 = \tfrac{1}{18}.$$

Consider first the expressions for p' and w'. The term of lowest degree in p' is, by (29), $p' = p'_4 m^4 = EMN_2 m^4/6(E+M)^2$. It follows from (51) that $p'_4 m^4 = 1/80{,}000{,}000$. That is, since $r' = a'(1 + p')$, the average distance of the sun from the center of gravity of the earth and moon is about 1.2 miles greater than it would be if it were not disturbed by the moon. The first variable terms in p' and w' are, by (30),

$$p'_6 m^6 = +\tfrac{3}{8}\frac{EMN_2 m^6}{(E+M)^2}\cos 2\tau, \qquad w'_6 m^6 = -\tfrac{3}{8}\frac{EMN_2 m^6}{(E+M)^2}\sin 2\tau$$

Therefore, with respect to the center of gravity of the earth and moon as an origin, the sun describes in a period of one half a synodical month in the retrograde direction a nearly circular orbit of radius 95 feet around a point whose distance is approximately $a'(1 + p'_4 m^4 + p'_5 m^5 + p'_6 m^6)$, and which revolves with the mean angular motion of the sun.

The coördinates of the sun, p' and w', enter in the right members of the first two equations of (9). It has been shown how slightly they vary because of perturbations by the moon. The reciprocal action of these perturbations upon the moon are still smaller, because where they appear in the differential equations they are multiplied by m^2. If we regard $MN_2/(E+M)$ as of the second order in m, the perturbations of the moon by the sun due to the fact that the moon has disturbed the sun from a strictly circular orbit are of the 8th order in m, and their numerical coefficients are so small that they are absolutely inappreciable observationally. That is, no sensible error is committed in computing the moon's coördinates by assuming that the sun goes around the center of gravity of the earth and moon in an undisturbed orbit.

It is seen from (51) that the constant η in the Lunar Theory is very nearly equal to unity. Since the higher terms are much simplified by replacing it by unity while no sensible error is committed, we shall retain it explicitly only up to terms of order 4 in m. By applying equations (42) where the $A_j^{(i)}$ and $B_j^{(i)}$ are defined in (32), it has been found that *

$$\begin{aligned}\frac{r}{a} = &-\left[\frac{1}{6}+\cos 2\tau\right]\eta m^2+\left[\frac{1}{3}+\frac{15}{16}M_1\cos\tau-\frac{7}{6}\cos 2\tau\right]\eta m^3+\left[\frac{331}{32\cdot 9}\eta-\frac{1}{2}\right.\\ &+\frac{3}{32}(45\eta-1)M_1\cos\tau+\frac{2}{9}(3\eta-5)\cos 2\tau-\frac{25}{64}M_1\cos 3\tau\\ &\left.-\frac{3}{8}\eta\cos 4\tau\right]\eta m^4+\left[\frac{181}{16\cdot 9}+\frac{3145}{256}M_1\cos\tau-\frac{19}{2\cdot 27}\cos 2\tau+\frac{95}{128}M_1\cos 3\tau\right.\\ &\left.-\frac{497}{32\cdot 3\cdot 5}\cos 4\tau\right]m^5+\left[\left(\frac{3175}{32\cdot 81}+\frac{225}{512}M_1^2-\frac{3}{16}M_2\right)+\frac{56831}{1024}M_1\cos\tau\right.\\ &-\left(\frac{10933}{256\cdot 81}+\frac{225}{512}M_1^2+\frac{5}{8}M_2\right)\cos 2\tau+\frac{5053}{256\cdot 3}M_1\cos 3\tau-\left(\frac{4897}{16\cdot 9\cdot 25}\right.\\ &\left.\left.+\frac{7}{32}M_2\right)\cos 4\tau-\frac{155}{512}M_1\cos 5\tau-\frac{59}{256}\cos 6\tau\right]m^6+\left[\left(\frac{245}{8\cdot 27}+\frac{105}{32}M_1^2\right.\right.\\ &\left.+\frac{3}{8}M_2\right)+\left(\frac{910{,}469}{1024\cdot 3}M_1+\frac{105}{128}M_3\right)\cos\tau+\left(\frac{911{,}671}{512\cdot 243\cdot 5}-\frac{8985}{1024}M_1^2\right.\\ &\left.-\frac{5}{8}M_2\right)\cos 2\tau+\frac{285{,}283}{4096\cdot 3}M_1\cos 3\tau-\left(\frac{130{,}723}{32\cdot 27\cdot 125}-\frac{525}{1024}M_1^2\right.\\ &\left.\left.+\frac{49}{32\cdot 3\cdot 5}M_2\right)\cos 4\tau+\frac{625}{1024\cdot 3}M_1\cos 5\tau-\frac{51853}{512\cdot 3\cdot 5\cdot 7}\cos 6\tau\right]m^7\end{aligned}$$

* Mr. W. D. MacMillan has duplicated the entire computation as a check. Moreover, the integral and other check formulas have been exactly satisfied, and it is not believed that there can be any errors in these series.

$$+\left[\left(\frac{2,228,365}{8192\cdot 243}+\frac{165,615}{8192}M_1^2+\frac{53}{32}M_2\right)+\left(\frac{228,305,129}{16384\cdot 9}M_1+\frac{1125}{256}M_1M_2\right.\right.$$

$$\left.+\frac{225}{64}M_3\right)\cos\tau+\left(\frac{8,581,859}{64\cdot 729\cdot 25}-\frac{203,593}{4096}M_1^2+\frac{1681}{256}M_2\right)\cos 2\tau$$

$$+\left(\frac{1,145,073}{4096\cdot 3}M_1-\frac{245}{1024}M_3\right)\cos 3\tau-\left(\frac{8,398,553}{256\cdot 81\cdot 625}-\frac{4245}{2048}M_1^2\right.$$

$$\left.+\frac{3241}{64\cdot 9\cdot 25}M_2\right)\cos 4\tau+\left(\frac{943,835}{16384\cdot 9}M_1-\frac{147}{1024}M_3\right)\cos 5\tau$$

$$-\left(\frac{698,591}{32\cdot 9\cdot 25\cdot 49}+\frac{425}{8192}M_1^2+\frac{49}{256}M_2\right)\cos 6\tau-\frac{31885}{16384\cdot 7}M_1\cos 7\tau$$

$$\left.-\frac{529}{1024\cdot 3}\cos 8\tau\right]m^8+\left[\left(-\frac{13,799}{4096\cdot 9}+\frac{5,765,369}{16384\cdot 3}M_1^2+\frac{5}{4\cdot 3}M_2\right)\right.$$

$$+\left(\frac{4,337,803,547}{65536\cdot 9}M_1+\frac{7575}{256}M_1M_2+\frac{64,025}{2048\cdot 3}M_3+\frac{166,875}{32768}M_1^3\right)\cos\tau$$

$$+\left(\frac{1,924,392,199}{512\cdot 2187\cdot 125}-\frac{968,633}{512\cdot 9}M_1^2+\frac{346,881}{512\cdot 9\cdot 5}M_2\right)\cos 2\tau$$

$$+\left(\frac{1,018,308,773}{16384\cdot 27\cdot 5}M_1-\frac{14,445}{8192}M_1M_2+\frac{3045}{4096}M_3+\frac{10,125}{32768}M_1^3\right)\cos 3\tau$$

$$+\left(\frac{1,092,030,283}{1024\cdot 27\cdot 3125\cdot 7}-\frac{3,348,655}{16384\cdot 9\cdot 5}M_1^2-\frac{259,997}{64\cdot 27\cdot 125}M_2\right)\cos 4\tau$$

$$+\left(\frac{264,181,703}{65536\cdot 27\cdot 5}M_1+\frac{3045}{8192}M_1M_2-\frac{217}{4096}M_3\right)\cos 5\tau$$

$$+\left(-\frac{1,597,008,893}{512\cdot 27\cdot 125\cdot 343}+\frac{62,915}{4096\cdot 3\cdot 7}M_1^2-\frac{3271}{512\cdot 3\cdot 5}M_2\right)\cos 6\tau$$

$$\left.-\frac{5057}{32768}M_1\cos 7\tau-\frac{5,778,113}{8192\cdot 3\cdot 5\cdot 49}\cos 8\tau\right]m^9+\cdots,$$

$$v=(1+m)\tau+\left[\frac{11}{8}\sin 2\tau\right]\eta m^2+\left[-\frac{15}{8}M_1\sin\tau+\frac{13}{6}\sin 2\tau\right]\eta m^3$$

$$+\left[-\frac{3}{16}(45\eta+7)M_1\sin\tau-\frac{1}{18}(9\eta-41)\sin 2\tau+\frac{15}{32}M_1\sin 3\tau+\frac{201}{256}\sin 4\tau\right]\eta m^4$$

$$+\left[-\frac{4151}{128}M_1\sin\tau+\frac{89}{4\cdot 27}\sin 2\tau-\frac{25}{16}M_1\sin 3\tau+\frac{2177}{64\cdot 3\cdot 5}\sin 4\tau\right]m^5$$

$$+\left[-\frac{71,449}{512}M_1\sin\tau-\left(\frac{62,911}{2048\cdot 81}-\frac{1125}{1024}M_1^2-\frac{25}{32}M_2\right)\sin 2\tau\right.$$

$$-\frac{413}{32}M_1\sin 3\tau+\left(\frac{23,737}{32\cdot 9\cdot 25}+\frac{63}{256}M_2\right)\sin 4\tau+\frac{75}{128}M_1\sin 5\tau$$

$$\left.+\frac{3715}{2048\cdot 3}\sin 6\tau\right]m^6+\left[-\left(\frac{2,190,245}{1024\cdot 3}M_1+\frac{105}{64}M_3\right)\sin\tau+\left(\frac{-578,077}{128\cdot 243\cdot 5}\right.\right.$$

$$\left.+\frac{34,155}{2048}M_1^2+\frac{5}{4}M_2\right)\sin 2\tau-\frac{100,995}{2048}M_1\sin 3\tau+\left(\frac{1,339,541}{128\cdot 27\cdot 125}\right.$$

$$\left.\left.-\frac{1725}{2048}M_1^2+\frac{77}{32\cdot 3\cdot 5}M_2\right)\sin 4\tau-\frac{919}{1024}M_1\sin 5\tau+\frac{4289}{16\cdot 3\cdot 5\cdot 7}\sin 6\tau\right]m^7$$

$$+\left[-\left(\frac{90,908,791}{8192\cdot 3}M_1+\frac{1125}{128}M_1M_2+\frac{135}{16}M_3\right)\sin\tau-\left(\frac{54,988,283}{256\cdot 729\cdot 25}\right.\right.$$

$$\left.-\frac{811,967}{8192}M_1^2+\frac{45,721}{2048\cdot 3}M_2\right)\sin 2\tau-\left(\frac{7,378,645}{4096\cdot 9}M_1-\frac{385}{512\cdot 3}M_3\right)\sin 3\tau$$

$$+\left(\frac{652,081,583}{8192\cdot 81\cdot 625}-\frac{10,125}{4096}M_1^2+\frac{91,069}{512\cdot 9\cdot 25}M_2\right)\sin 4\tau-\left(\frac{133,763}{8192}M_1\right.$$

$$\left.-\frac{399}{512\cdot 5}M_3\right)\sin 5\tau+\left(\frac{338,729}{256\cdot 5\cdot 49}+\frac{1675}{16384}M_1^2+\frac{693}{2048}M_2\right)\sin 6\tau$$

$$\left.+\frac{279,055}{8192\cdot 49}M_1\sin 7\tau+\frac{207,155}{131072\cdot 3}\sin 8\tau\right]m^8+\left[\left(-\frac{15,772,500,205}{32768\cdot 27}M_1\right.\right.$$

$$\left.-\frac{17,415}{256}M_1M_2-\frac{84,575}{1024\cdot 3}M_3-\frac{173,625}{16384}M_1^3\right)\sin\tau+\left(-\frac{1,601,782,811}{256\cdot 2187\cdot 125}\right.$$

$$\left.+\frac{4198}{9}M_1^2-\frac{182,881}{512\cdot 3\cdot 5}M_2\right)\sin 2\tau+\left(-\frac{537,179,239}{4096\cdot 27\cdot 5}M_1+\frac{6075}{4096}M_1M_2\right.$$

$$\left.-\frac{9205}{2048\cdot 3}M_3-\frac{14,625}{16384}M_1^3\right)\sin 3\tau+\left(-\frac{3,297,305,351}{512\cdot 81\cdot 3125\cdot 7}\right.$$

$$\left.+\frac{5,397,281}{32768\cdot 9}M_1^2+\frac{1,097,537}{128\cdot 27\cdot 125}M_2\right)\sin 4\tau+\left(-\frac{22,567,883}{32768\cdot 9}M_1\right.$$

$$\left.-\frac{2205}{4096}M_1M_2+\frac{161}{2048}M_3\right)\sin 5\tau+\left(\frac{1,673,061,443}{64\cdot 81\cdot 125\cdot 343}-\frac{195,865}{16384\cdot 7}M_1^2\right.$$

$$\left.+\frac{5659}{512\cdot 3\cdot 5}M_2\right)\sin 6\tau-\frac{6857}{8192\cdot 3\cdot 7}M_1\sin 7\tau$$

$$\left.+\frac{70,605,533}{32768\cdot 3\cdot 5\cdot 49}\sin 8\tau\right]m^9+\cdots.$$

If in these series η is put equal to unity the terms which are independent of M_1, M_2, and M_3 correspond to those found by HILL in his researches on the Lunar Theory.* It is not easy to compare this work with HILL's for he used rectangular coördinates, and in the final series a different parameter. Moreover, his linear constant was different from the a defined in (4) and used here. With $\eta = 1$ the terms involving M_1, M_2, and M_3 correspond to those found by BROWN.† The difficulty of comparing with his results is the same as that of comparing with Hill's work.

DELAUNAY used the reciprocal of the radius vector, the longitude, and the latitude as coördinates in his work on the Lunar Theory. Instead of the parameter m he used in its place the ratio of the mean motion of the sun to that of the moon. If we call it μ, it is

$$\mu = \frac{n'}{n}, \text{ whence } m = \frac{\mu}{1-\mu}.$$

DELAUNAY and other investigators in the Lunar Theory have used Fourier series whose coefficients are power series in the parameters employed. If we rearrange the series above as Fourier series and transform from m to μ, we may easily compare with DELAUNAY's expression for the longitude. In order to make the comparison we must, in the series above, put

$$\eta = 1, \qquad \frac{E^j - (-1)^j M^j}{(E+M)^j} = 1, \qquad m^{2j} M_j = \left(\frac{a}{a'}\right)^j.$$

The $\sin j\tau$ above is $\sin jD$ in DELAUNAY's notation. Perfect agreement is found in the coefficients of all terms except the following:

* American Journal of Mathematics, vol. 1, p. 143.

† American Journal of Mathematics, vol. 14, pp. 140–160.

Coefficient of	Delaunay's value.	Present computation.
$\frac{a}{a'}\mu^6 \sin \tau$	$-\frac{63,106,813}{8192}$	$-\frac{74,009,397}{8192}$
$\frac{a}{a'}\mu^7 \sin \tau$	$-\frac{10,835,537,159}{65536 \cdot 3}$	$-\frac{47,823,726,517}{32768 \cdot 27}$
$\left(\frac{a}{a'}\right)^3 \mu^2 \sin \tau$	not computed	$-\frac{2415}{128}$
$\left(\frac{a}{a'}\right)^3 \mu^3 \sin \tau$	" "	$-\frac{6,981,835}{16384 \cdot 3}$
$\mu^8 \sin 2\tau$	$+\frac{7,596,606,727}{4096 \cdot 729 \cdot 25}$	$+\frac{3,748,175,501}{2048 \cdot 729 \cdot 25}$
$\mu^9 \sin 2\tau$	$-\frac{8,051,418,161}{2048 \cdot 2187 \cdot 125}$	$-\frac{475,281,673}{128 \cdot 2187 \cdot 125}$
$\left(\frac{a}{a'}\right)^2 \mu^4 \sin 2\tau$	not computed	$+\frac{3,713,357}{8192 \cdot 3}$
$\left(\frac{a}{a'}\right)^2 \mu^5 \sin 2\tau$	" "	$+\frac{85,194,253}{2048 \cdot 9 \cdot 5}$
$\frac{a}{a'}\mu^6 \sin 3\tau$	" "	$-\frac{21,715,555}{4096 \cdot 9}$
$\frac{a}{a'}\mu^7 \sin 3\tau$	" "	$-\frac{1,764,511,039}{4096 \cdot 27 \cdot 5}$
$\left(\frac{a}{a'}\right)^3 \mu^2 \sin 3\tau$	" "	$+\frac{385}{512 \cdot 3}$
$\left(\frac{a}{a'}\right)^3 \mu^3 \sin 3\tau$	" "	$-\frac{19,975}{16384 \cdot 3}$
$\mu^8 \sin 4\tau$	$+\frac{123,030,377,303}{8192 \cdot 81 \cdot 625}$	$+\frac{82,679,012,303}{8192 \cdot 81 \cdot 625}$
$\mu^9 \sin 4\tau$	not computed	$+\frac{876,528,810,303}{1024 \cdot 81 \cdot 3125 \cdot 7}$
$\left(\frac{a}{a'}\right)^2 \mu^4 \sin 4\tau$	" "	$-\frac{2,754,403}{4096 \cdot 9 \cdot 25}$
$\left(\frac{a}{a'}\right)^2 \mu^5 \sin 4\tau$	" "	$-\frac{1,217,563,507}{32768 \cdot 27 \cdot 125}$
$\frac{a}{a'}\mu^5 \sin 5\tau$	$+\frac{3911}{1024}$	$+\frac{1481}{1024}$
$\frac{a}{a'}\mu^6 \sin 5\tau$	not computed	$-\frac{122,523}{8192}$
$\frac{a}{a'}\mu^7 \sin 5\tau$	" "	$-\frac{51,974,771}{32768 \cdot 9}$

Coefficient of	Delaunay's value.	Present computation.
$\left(\frac{a}{a'}\right)^3 \mu^2 \sin 5\tau$	not computed	$+\frac{399}{512 \cdot 5}$
$\left(\frac{a}{a'}\right)^3 \mu^3 \sin 5\tau$	" "	$+\frac{1309}{4096}$
$\mu^8 \sin 6\tau$	" "	$+\frac{54,143,779}{2048 \cdot 3 \cdot 5 \cdot 49}$
$\mu^9 \sin 6\tau$	" "	$+\frac{138,760,838,897}{256 \cdot 81 \cdot 125 \cdot 343}$
$\left(\frac{a}{a'}\right)^2 \mu^4 \sin 6\tau$	" "	$+\frac{7219}{16384}$
$\left(\frac{a}{a'}\right)^2 \mu^5 \sin 6\tau$	" "	$-\frac{1,361,621}{16384 \cdot 3 \cdot 5 \cdot 7}$
$\frac{a}{a'} \mu^6 \sin 7\tau$	" "	$+\frac{279,055}{8192 \cdot 49}$
$\frac{a}{a'} \mu^7 \sin 7\tau$	" "	$+\frac{4,974,991}{8192 \cdot 3 \cdot 49}$
$\mu^8 \sin 8\tau$	" "	$+\frac{207,155}{131072 \cdot 3}$
$\mu^9 \sin 8\tau$	" "	$+\frac{34,667,316,703}{32768 \cdot 3 \cdot 5 \cdot 49}$

The very general agreement shows that DELAUNAY's terms involving the argument jD belong strictly to a periodic orbit, and that he carried out the computations with remarkable accuracy * considering the complicated operations which his method demanded.

When the expressions for the coördinates are arranged as Fourier series, it is found that the coefficients expressed in terms of m converge much more rapidly, at least to the ninth power, than when expressed in terms of μ. HILL showed † that if a parameter m' is defined by

$$m' = \frac{m}{1 - \frac{m}{3}},$$

the convergence is still more improved. This transformation removes the pole arising in the expansion of the principal term from $|m| = \sqrt{6}$ to $|m'| = 3$. In μ the pole is at the distance $|\mu| = \sqrt{\frac{6}{11}}$. If an m'' is introduced by

* BROWN's results disagree with DELAUNAY's in the coefficients of $(a/a')\mu^5 \sin j$D; he did not compute higher powers. Loc. cit., p. 155.

† Loc. cit., p. 141.

$$m'' = \frac{m^2}{6 - 4m + m^2}$$

the pole of the expression is removed to infinity and the convergence of the series for the coefficients of $\cos 2\tau$ and $\sin 2\tau$, at least out to the 9th power, is slightly improved. But these coefficients are given by any of the series with sufficient accuracy. The slowest convergence is in the coefficient of $\sin \tau$. In DELAUNAY's parameter this coefficient is

$$\left[-\frac{15}{8}\mu - \frac{93}{8}\mu^2 - \frac{6887}{128}\mu^3 - \frac{137,197}{512}\mu^4 - \frac{4,628,333}{1024 \cdot 3}\mu^5 - \frac{74,009,397}{8192}\mu^6 - \frac{47,823,726,517}{32768 \cdot 27}\mu^7 \cdots \right] \begin{matrix} a \\ a' \end{matrix}.$$

In the parameter of this paper the same coefficient is

$$(52) \quad \left[-\frac{15}{8}m - \frac{39}{4}m^2 - \frac{4151}{128}m^3 - \frac{71,449}{512}m^4 - \frac{2,190,245}{1024 \cdot 3}m^5 - \frac{90,908,791}{8192 \cdot 3}m^6 - \frac{15,772,500,205}{32768 \cdot 27}m^7 \cdots \right] \begin{matrix} a \\ a' \end{matrix}$$

When expressed in seconds the coefficient of the term containing the 6th and 7th powers of the parameter is $-0''.7122$ and $-0''.3811$ in DELAUNAY's series, and $-0''.5453$ and $-0''.2124$ in the series given here. While the two series represent the same function, the first seven terms in m seem to give it more accurately than the first seven terms in μ.

It is noticed that the numerical coefficients so far as given are all of the same sign, and that the ratio of each coefficient to the preceding one seems to tend approximately toward 5. If this were the precise ratio, and if it were indefinitely maintained, the series would be simply the expansion of $(1 - 5m)^{-1}$. If the first few terms succeeding those which have been computed proceed approximately according to the same law, then the function will be much more accurately given by them if they are multiplied and divided by $1 - 5m$. Thus we find by these operations in place of (52)

$$(53) \quad \left[\frac{-\frac{15}{8}m - \frac{3}{8}m^2 + \frac{2089}{128}m^3 + \frac{11,571}{512}m^4 - \frac{46,775}{1024 \cdot 3}m^5 - \frac{3,298,991}{3 \cdot 8192}m^6 + \frac{591,082,175}{32768 \cdot 27}m^7 \cdots}{1 - 5m} \right] \begin{matrix} a \\ a' \end{matrix}.$$

In applying HILL's method to the *parallactic inequalities* BROWN found* that

* Loc. cit., p. 149.

the divisor $1 - 4m - \frac{37}{8}m^2 \cdots$ naturally enters. The terms in the series with this divisor decrease somewhat less rapidly than those in (53). It is to be noted that the radius of convergence is never altered by using any finite polynomial as a multiplier and divisor in this way. The process is in reality a sort of method of exterpolating which is useful if the coefficients proceed for a time according to the assumed law.

The average ratio of each coefficient to the preceding one for the six terms given in (52) is approximately $\frac{23}{5}$. Hence we may conveniently write the series as $-\frac{15}{8}m\left[1 - \frac{23}{5}m\right]^{-1}$ plus a series which shall make, when added to the expansion of this function, the six terms given in (52). We find in place of (52)

$$(54)\quad \left[-\frac{15}{8}m\left[1-\frac{23}{5}m\right]^{-1} - \frac{9}{8}m^2 + \frac{4637}{128\cdot 5}m^3 + \frac{549{,}839}{512\cdot 25}m^4 + \frac{48{,}596{,}207}{1024\cdot 3\cdot 125}m^5 + \frac{2{,}499{,}342{,}713}{8192\cdot 3\cdot 625}m^6 - \frac{174{,}308{,}031{,}761}{32768\cdot 27\cdot 3125}m^7 + \cdots\right]\frac{a}{a'}.$$

To illustrate the effects of these processes we shall reduce these series to numbers. It has been found from the observations that

$$m = 0.08084893, \qquad \frac{a}{a'} = 0.00255878.$$

With these numbers we get for (52), (53), and (54) respectively

$$-80''.008 - 33''.637 - 9''.045 - 3''.147 - 1''.300 - 0''.545 - 0''.212 \cdots = -127''.894,$$

$$\frac{-80''.008 - 1''.294 + 4''.552 + 0''.516 - 0''.028 - 0''.020 + 0''.008 \cdots}{0.59575535} = -128''.029,$$

$$-127''.382 - 3''.882 + 2''.021 + 0''.969 + 0''.231 + 0''.024 - 0''.001 \cdots = -128''.020.$$

The corresponding values in DELAUNAY's series including a term of the 7th degree in μ are

$$-74''.024 - 34''.330 - 11''.885 - 4''.428 - 1''.862 - 0''.712 - 0''.381 \cdots = -127''.622.$$

BROWN found * for this coefficient, using the numerical value of m from the outset, $-128''.069$. While this number is slightly larger than one would infer

* Monthly Notices of R. A. S., vol. 64 (1904), 534.

from (52), (53), or (54), it is very probably essentially correct. But in the computation we put $\eta = 1$, $M_1 m^2 = a/a'$, whereas the correct values are

$$\eta = \frac{1}{1 + \dfrac{E+M}{S'}} = 0.999997,$$

$$M_1 m^2 = \frac{E-M}{E+M}\frac{a}{a'} = \frac{80.487}{82.487}\frac{a}{a'}.$$

With these values of the constants the coefficient (53) is $-124''.925$. HILL has computed† the coefficients of the various arguments of the series with a very high degree of accuracy by substituting their approximate values as determined by himself and Brown in the differential equations, and giving them such differential corrections that they more exactly satisfy the equations. He found for the coefficient of $\sin \tau$, when we change from his value of a/a' to that used here, $-124''.965$.

6. Applications to Darwin's periodic orbits.

In his memoir in *Acta Mathematica*, vol. 21, pp. 99–242, DARWIN considered two finite bodies of masses 10 and 1 revolving around their center of gravity in circles, and an infinitesimal body moving in the plane of motion of the finite bodies and subject only to their attraction. He discovered a large number of periodic orbits by interpolation from series of special orbits computed by mechanical quadratures. It will be shown that his "Satellites A" and "Planets A" belong to the type of orbits treated in this paper.

To make the series computed in this paper applicable to the satellite orbits treated by DARWIN it is necessary to put $M = 0$, $E = 1$, $S = 10$, $a' = 1$. Then, when the infinitesimal body moves around E, we have from (4) and (6),

$$(55) \qquad \eta = \frac{S}{E+S} = \frac{10}{11},$$

$$\frac{a}{a'} = \left(\frac{n'}{n}\right)^{\frac{2}{3}}\left(\frac{E}{E+S}\right)^{\frac{1}{3}} = \left(\frac{1}{11}\right)^{\frac{1}{3}}\left(\frac{m}{1+m}\right)^{\frac{2}{3}}.$$

DARWIN defined his orbits by the value of the Jacobian constant, and their periods were found from the detailed computations. The Jacobian integral may be written

$$F = F_0 + F_1 m + F_2 m^2 + \cdots = C.$$

For an arbitrary value of m sufficiently small, as m_0, the value of C can be computed. Let it be C_0; then

† Astronomical Journal, vol. 15 (189), p. 137.

$$F_0 + F_1 m_0 + F_2 m_0^2 + \cdots = C_0.$$

Let the value of C of the required orbit be $C = C_0 + \Delta C$, and the corresponding value of m be $m = m_0 + \Delta m$. Substituting these values in the integral and inverting the series, we have Δm expressed in terms of ΔC, and therefore m expressed in terms of C. With this value of m the coördinates can be computed from the series which have been given above.

While there is no difficulty in carrying out the details of the steps just described, it will be sufficient for the comparison to take the period as found by the computation. We shall compare with "Satellite A" (Jacobian constant $= 40.5$), *loc. cit.* p. 199. The synodic period was found to be $61° 23' = 61°.383$, where the period of the finite bodies is 360°. Therefore

$$m = \frac{61.383}{360} = \frac{1}{5.8648} = 0.17051,$$

$$\frac{a}{a'} = \left(\frac{1}{11}\right)^{\frac{1}{3}} \left(\frac{m}{1+m}\right)^{\frac{1}{3}} = 0.12449.$$

The m is more than twice as large as that occurring in the Lunar Theory, while the a/a' is more than 40 times as great as a/a' for the moon and sun.

Putting $M_j m^{2j} = (a/a')^j$ in the series, and using these values of the constants we find

$$r = 0.12427 + 0.00652 \cos \tau - 0.00420 \cos 2\tau + 0.00004 \cos 3\tau - 0.00006 \cos 4\tau \cdots,$$

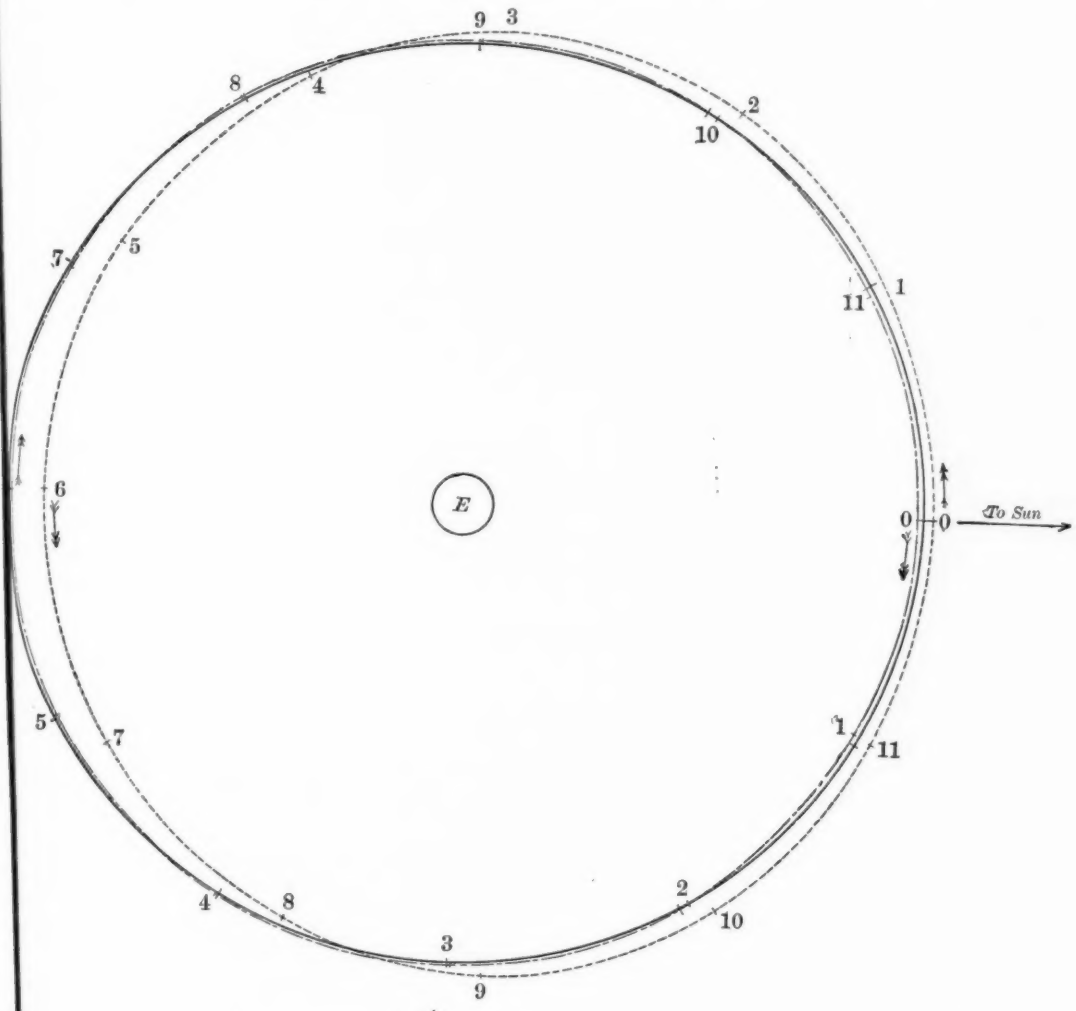
$$w = -0.12062 \sin \tau + 0.05079 \sin 2\tau - 0.00184 \sin 3\tau + 0.00095 \sin 4\tau \cdots.$$

The infinitesimal body is in a line with the finite bodies and between them when $\tau = 0$. The value of r at this time is $r(0) = 0.12657$. The corresponding value given by DARWIN is 0.1265, an agreement within the limits of accuracy of his computations. The infinitesimal body is in opposition at $\tau = \pi$, and we find for the radius at this time $r(\pi) = 0.11345$. DARWIN's value is 0.1135. The agreements are sufficient to show that the analysis of this paper pertains to DARWIN's "Satellites A."

To get the coördinates in a retrograde orbit having the same sidereal period, it is necessary to change the sign of n in $m = n'/(n - n')$. In the example just treated it is found that

$$\frac{n'}{n} = \frac{61.383}{421.383},$$

whence m in the retrograde orbit having the same sidereal period is



$$m = \frac{-\frac{n'}{n}}{1+\frac{n'}{n}} = -\frac{61.383}{482.766} = \frac{-1}{7.8648}.$$

The value of a/a' is the same as before and the series give for the coördinates

$$r = +0.12412 - 0.00057\cos\tau - 0.00158\cos 2\tau$$
$$- 0.00005\cos 3\tau - 0.00001\cos 4\tau + \cdots,$$

$$w = +0.00820 \sin \tau + 0.01654 \sin 2\tau + 0.00047 \sin 3\tau + 0.00010 \sin 4\tau + \cdots.$$

These two orbits and the circular orbit of which they are the analytic continuation, together with the positions at intervals of $\frac{1}{12}$ of the period, are given in the figure.

DARWIN found periodic planetary orbits corresponding to the "Satellites A." The analysis above gives these orbits within the limits of the convergence of the series if we put $E = 10$, $S = 1$, $M = 0$, $a' = 1$, $\eta = S/(E + S') = \frac{1}{11}$. For the comparison with an actual example we shall take "Planet A" for the Jacobian constant equal to 40.0 (loc. cit., p. 225). The period given by DARWIN's computation is $nT = 154° 13'$, whence

$$m = \frac{154.216}{360} = \frac{1}{2.3344} = 0.42838, \qquad \frac{a}{a'} = 0.43404, \qquad M_1 = 2.3652.$$

With these values of the parameters the series for the radius becomes

$$r = 0.43373 + 0.00776 \cos \tau - 0.01286 \cos 2\tau - 0.00104 \cos 3\tau \cdots.$$

When the infinitesimal body is between E and S, and when it is in opposition to S, the values of r are respectively $r = 0.42759$ and $r = 0.41415$. DARWIN found in these respective cases $r = 0.423$ and $r = 0.4140$. These results agree within the limits of error of the computations and show that the "Planets A" belong to the type of orbits covered by the analysis of this paper.

DARWIN's "Satellites B" and "Satellites C" do not exist until the Jacobian constant has diminished nearly to 39.5. At first appearance they are coincident, but with decreasing values of the Jacobian constant they gradually diverge. The computations of DARWIN show that they are not isolated orbits existing only for special values of the parameter m. Suppose one of these orbits is known for $m = m_0$. Unless the orbits "B" and "C" coincide for this value of m, the coördinates of an orbit near the known one may be expressed as power series in $m - m_0 = \lambda$; if the orbits "B" and "C" coincide for this value of m, the coördinates of the two neighboring orbits may be expressed as power series in $\sqrt{\lambda}$ and $-\sqrt{\lambda}$ respectively. When λ has one sign their coördinates are real, and when it has the other, they are conjugate complex quantities. Suppose the series converge so long as $|\lambda| < \lambda_0$. The coördinates of the orbits near the ones belonging to $m = m_0 + \lambda$ may be expanded as power series in $m - m_0 - \lambda_0 = \lambda'$. If the complex orbits are in question there will be two, one with λ and one with $-\lambda$, whose coördinates will still be complex and conjugate. They have the property of reëntering after one synodic revolution.

Suppose the two orbits which branch at $m = m_0$ are followed until $m_0 + \lambda_0 + \cdots = 0$. Their coördinates will remain conjugate through the whole series of steps, and the property of reëntering after one synodic revolution will

be preserved. But for this value of m the disturbance vanishes and the differential equations reduce to those which belong to the two body problem. The period remains real in t as well as in τ. Consequently these orbits show that there are at least two conjugate complex solutions of the problem of two bodies whose period in t is real.

Let us consider the differential equations expressed in t. From (6) we have

$$\tau = \frac{n'}{m} t = n\left(1 - \frac{n'}{n}\right) t,$$

and the differential equations may be expressed in terms of $n'/n = \mu$. The parameter enters in two ways, (a) independently of the trigonometric functions, and (b) under the trigonometric functions. We shall generalize the differential equations by distinguishing the μ according as it enters in one way or the other. Where it enters under the trigonometric functions it will be regarded as an assigned number which admits of no change in the development of the solutions; where it enters in the other way it will be regarded as a parameter capable of taking all values sufficiently small, and it will be made to play the same rôle as m has in the preceding work. The solutions exist for all values of μ sufficiently small, but they belong to the physical problem only when the parameter μ is set equal to the μ which occurs under the trigonometrical functions. The series of orbits is obtained by assigning a series of values to the μ in the trigonometric functions.

In the orbits in question the angular motion of the finite bodies is real; therefore n' is real. Also the period of the infinitesimal body is real; therefore μ and n are real. The orbit of the finite bodies is real; therefore a' is real and equal to unity with DARWIN's choice of the linear unit. The parameter a is subjected only to the relation (7)

$$a = a'\left(\frac{m}{1+m}\right)^{\frac{2}{3}}\left(\frac{E}{E+S}\right)^{\frac{1}{3}} \qquad (M \text{ being infinitesimal}).$$

There are precisely three values of a satisfying this relation, one of which is real while the other two are conjugate complex quantities. All three satisfy the conditions of the two body problem equally. Likewise the periodic solutions of the three body problem exist for all three roots for all values of μ sufficiently small. Moreover these are the only solutions of the type under consideration in which the periods are real. In the three cases the coefficients of all powers of a/a' are precisely the same, and since the three values of a/a' have the same modulus, the series have the same radius of convergence. Since these are the only solutions of the type in question, it follows that DARWIN's orbits "Satellites B" and "Satellites C" are the analytic continuation of these complex orbits, and that his search for the satellite orbits was exhaustive. His work

shows the interesting and important result that the two imaginary orbits unite and become real (if the distribution of masses is 10 and 1) when m is a little less than $\frac{1}{4}$.

The remaining problem respecting the series which have been developed is their true radius of convergence for a given distribution of mass between the finite bodies. For simplicity suppose the coördinates and components of velocity of the infinitesimal body are $x_1, \cdots, x_4$. Then the differential equations may be written

$$\frac{dx_j}{dt} = f_j(x_1, \cdots, x_4, \mu; t) \qquad (j=1, \cdots, 4).$$

The initial values of the x_j are $\alpha, e, 0, 0$. Then the solutions may be written

$$x_j = \mathfrak{p}_j(\alpha, e, \mu; t), \tag{56}$$

which converge for any preassigned interval in t if the moduli of α, e and m are sufficiently small. HAPPEL* has shown that they converge for the values of these parameters which belong to the moon's variational orbit. But this by no means proves the convergence of the series in μ which belong to the variational orbit. The parameters α and e are determined in terms of μ by the periodicity conditions (on two of the equations for symmetrical orbits)

$$p_i\left(\alpha, e, \mu; \frac{T}{2}\right) = 0, \qquad p_j\left(\alpha, e, \mu; \frac{T}{2}\right) = 0, \tag{57}$$

where T is the period. These series are to be solved for α and e as power series in μ and the results substituted in (56). In general, new singularities are introduced by the conditions (57) and are carried back into (56). Consequently in discussing the convergency of the series for the variational orbit both the series (56) and the singularities introduced by (57) must be considered.

Consider equations (57). For that value of μ for which two orbits coincide these equations have a double solution, and this value of μ is a branch point of their solutions. Therefore the periodic solutions in μ do not converge for values of μ whose moduli are greater than the modulus of that value of μ for which two of the orbits coincide. Applying this to DARWIN'S computations, we find that the series for the complex orbits "B" and "C" do not converge for values of μ beyond which these orbits unite and become real. The series for the orbit "A" have the same radius of convergence as those for the complex orbits. Therefore the series for "A" do not converge for values of μ greater than that for which "B" and "C" become real. Expressed in m, the series do not converge for $|m| > \frac{1}{4}$. In the orbit computed $m = 1/5.86$, and an examination of the coefficients shows that this is near the limit of practical useful-

* *Inaugural Dissertation*, Göttingen, 1900.

ness of the series. Many of DARWIN's orbits belong to values of m for which the series diverge.

Unfortunately it cannot be positively asserted that the series converge until the orbits "B" and "C" become real. In the first place the parameter μ must be considered as complex except as it appears under the trigonometric functions, where it must remain real. (The radius of convergence is entirely independent of the value of the parameter under the sine and cosine functions.) Now it might happen that there would be a branching of orbits for a smaller modulus of μ in this complex realm. Such computations as those made by DARWIN would fail to show it, for the orbits, real or imaginary, belong to the physical problem only when μ, as the argument of the series, has the same value as occurs under the trigonometric functions.

The functions may branch in another way. An orbit which in general reënters only after two synodic revolutions may for a special value of μ coincide with one which reënters after one synodic revolution. At this point the orbit under consideration changes its stability.* DARWIN's "Satellite A" loses its stability for m about $1/3.7$. However, this seems not to have been because of a branching of the character in question, but because of a union of the orbit with the orbit of an "Oscillating Satellite a."* But there is no earlier change of stability, at least for real values of μ.

The series do not have a fixed radius of convergence as t goes through a period. The convergence in question is a uniform convergence with respect to t over the period of motion, and in this sense we have spoken of the series as having radii of convergence.

The problem of chief practical interest is the Lunar Theory. The distribution of mass between the finite bodies assumed by DARWIN, as compared with that occurring in the Lunar Theory, has decreased η a little and has increased a/a' for a given value of m about 30 fold. It seems probable that in the Lunar Theory the series converge for larger values of m than in DARWIN's problem, and the computation seems to show that in the latter problem they converge for $m = \frac{1}{6}$. Although it can scarcely be doubted that the series converge for the values of the parameters which the Lunar Theory requires, a proof of their convergence for these values is yet wanting. However, we do know that there is a limit within which the series are perfectly valid.

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* POINCARÉ's *Les Méthodes Nouvelles*, vol. 3, art. 380.

A PROBLEM IN THE REDUCTION OF HYPERELLIPTIC INTEGRALS*

BY

JOHN HECTOR McDONALD

The reduction of hyperelliptic integrals to elliptic has been treated in numerous memoirs. In the more general case of the reduction of abelian (non-hyperelliptic) integrals, the properties of the table of normal periods have been investigated by WEIERSTRASS † and M. POINCARÉ,‡ while the case $p = 3$ has been treated in detail by Mme. KOWALEWSKY † by WEIERSTRASS' methods.

Professor KÖNIGSBERGER § has drawn attention to the resemblance between JACOBI's transformation theory of elliptic integrals and the reduction theory of hyperelliptic integrals, which is such that the latter appears as a generalization of the former.

A fundamental memoir by M. PICARD ‖ contains new results and suggests new problems. It is shown that when, to the relation $y^2 = R_6(x)$, $R_6(x)$, denoting a polynomial of the sixth degree, there belongs a reducible integral, there exists also a second one, that these two form a system of normal integrals and that the table of periods corresponding to them is

$$\begin{matrix} 1 & 0 & G & \frac{1}{D} \\ 0 & 1 & \frac{1}{D} & G' \end{matrix} \qquad (D \text{ an integer}).$$

M. PICARD shows also that these integrals are both reducible by transformations of degree D. The condition for the existence of reducible integrals belonging to $y^2 = R_6(x)$ is an algebraic relation among the roots of $R_6(x) = 0$.

The following problem presents itself: Given one reducible integral and its reducing transformation, to determine the second one. This problem has given

* Presented to the Society (San Francisco) February 24, 1906. Received for publication July 11, 1906.

† Mme. KOWALEVSKY, Acta Mathematica, vol. 4 (1884), p. 393.

‡ POINCARÉ, Bulletin de la Société Mathématique de France, vol. 12 (1884), p. 124.

§ KÖNIGSBERGER, Crelle, vol. 67 (1867), p. 58, p. 97.

‖ PICARD, Bulletin de la Société Mathématique de France, vol. 11 (1883), p. 25.

rise to numerous researches. With the help of the theory of transformation of ϑ-functions, it has been solved when $D=4$ by Professor BOLZA.* By algebraic means, it has been solved when $D=3$ by M. GOURSAT,† and when $D=2$ by Professor HUTCHINSON.‡ The algebraic side has been studied by the writer§ when $D=4$. In the case $D=3$, an elegant geometrical investigation has been given by Professor BOLZA; ‖ while a portion of his results had been previously given by M. HUMBERT.¶

I propose in this paper to give the solution of the problem for the transformation of any order. The method which I shall use differs from any used by the writers mentioned and depends on an application of the theory of residues. Further the result will be so stated as to show its independence of linear transformation.

The reducing substitutions of reducible integrals may be studied by the methods that have been applied to the transforming substitutions of elliptic integrals. Further, there remains the problem of deriving the second reducible integral when one is given, while the hyperelliptic relation is not assumed to be in the normal form $y^2=R_6(x)$. To these problems I hope to return.

I. ALGEBRAIC FORMULATION OF THE PROBLEM.

Let $R_6(x)$ be a polynomial of the 6th degree and let $y^2=R_6(x)$. Then

$$\int \frac{P+Qx}{y}\,dx \tag{1}$$

is a hyperelliptic integral of genus 2 and of the first kind.

Such an integral is called reducible if there exists a rational function $f(x)/F(x)$ such that when the transformation of variable $z=f(x)/F(x)$ is made, the integral (1) is transformed into

$$\int \frac{dz}{\sqrt{R_4(z)}},$$

$R_4(z)$ being a polynomial of degree 4.

In order that the function $f(x)/F(x)$ shall exist, the polynomial $R_6(x)$ must satisfy a condition. The degree of this fraction is called the degree of the transformation.

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* BOLZA, Mathematische Annalen, vol. 28 (1887), p. 447.

† GOURSAT, Bulletin de la Société Mathématique de France, vol. 13 (1885), p. 143.

‡ HUTCHINSON, Chicago dissertation, 1897.

§ McDONALD, Transactions American Mathematical Society, vol. 2 (1901), p. 437.

‖ BOLZA, Mathematische Annalen, vol. 50 (1898), p. 313; vol. 51 (1899), p. 478.

¶ HUMBERT, American Journal of Mathematics, vol. 16 (1894), p. 238.

Supposing (1) to be reducible, we know from the theorem of M. PICARD that there exists a second integral

$$\int \frac{P' + Q'x}{y}\,dx \qquad (PQ' - P'Q \neq 0),$$

which is reducible by a transformation $z = \phi(x)/\Phi(x)$, and that the degrees of $f(x)/F(x)$ and $\phi(x)/\Phi(x)$ are the same.

The reduction problem may be put in an algebraic form from which appears the resemblance to JACOBI's transformation problem of elliptic integrals. Let homogeneous variables be introduced by writing $x = x_1/x_2$. The differential of x is then

$$\frac{x_2\,dx_1 - x_1\,dx_2}{x_2^2} \equiv \frac{(xdx)}{x_2^2}$$

in the current notation of the theory of invariants. The integral of the first kind appears as

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1x_2)}}, \qquad R_6(x_1x_2) \equiv (x\alpha)(x\beta)(x\gamma)(x\alpha')(x\beta')(x\gamma').$$

For reducibility we must be able to find two polynomials,

$$z_1 = f_1(x_1x_2), \qquad z_2 = f_2(x_1x_2)$$

of degree D such that the transformation of x into z gives

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1x_2)}} = \int \frac{(zdz)}{\sqrt{R_4(z_1z_2)}},$$

where

$$R_4(z_1z_2) = (z\lambda)(z\mu)(z\nu)(z\sigma).$$

In this form the problem of constructing reducible integrals or of determining whether or not a given integral is reducible is purely algebraic and is a special case of the construction of a binary involution of given discriminant, an important but unsolved problem.

On carrying out the transformation we obtain results analogous to those in the elliptic integral transformation theory.* We have to make a distinction between the cases D odd and D even. Supposing that $D = 2k + 1$ and letting H_{k-1}, ϕ_k, ψ_k, χ_k be forms of degree shown by their subscripts, we must have

$$\lambda_2 f_1 - \lambda_1 f_2 = (x\alpha)(x\beta)(x\gamma)\,H_{k-1}^2, \qquad \mu_2 f_1 - \mu_1 f_2 = (x\alpha')\,\phi_k^2,$$

$$\nu_2 f_1 - \nu_1 f_2 = (x\beta')\,\psi_k^2, \qquad \sigma_2 f_1 - \sigma_1 f_2 = (x\gamma')\,\chi_k^2,$$

$$(zdz) = \vartheta(xdx), \qquad \vartheta = H\phi\psi\chi(x\delta),$$

* JACOBI, *Fundamenta Nova*, *Werke*, vol. 1, p. 57.

where ϑ denotes the Jacobian of the forms f_1, f_2. We see then that $(x\delta)$, the numerator of the reducible integral, must be a factor of the Jacobian of f_1 and f_2, that is, a double element in the involution $\lambda f_1 + \mu f_2$.

Supposing that $D = 2k$, we must have (if the transformation is indecomposible)*

$$\lambda_2 f_1 - \lambda_1 f_2 = (x\alpha)(x\alpha')\phi_{k-1}^2, \qquad \mu_2 f_1 - \mu_1 f_2 = (x\beta)(x\beta')\psi_{k-1}^2,$$

$$\nu_2 f_1 - \nu_1 f_2 = (x\gamma)(x\gamma')\chi_{k-1}^2, \qquad \sigma_2 f_1 - \sigma_1 f_2 = H_k^2,$$

$$\vartheta = \phi\psi\chi H(x\delta), \qquad (zdz) = \vartheta(xdx).$$

Again $(x\delta)$ is a factor of the Jacobian of f_1, f_2.

The particular problem to be treated in this paper is the following: Given that

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1 x_2)}}$$

is reducible by the transformation

$$z_1 = f_1(x_1 x_2), \qquad z_2 = f_2(x_1 x_2),$$

to determine a linear form $(x\delta')$, so that

$$\int \frac{(x\delta')(xdx)}{\sqrt{R_6(x_1 x_2)}}$$

shall also be reducible.

II. The normal integrals and the reducing substitution.

Let us now return to non-homogeneous variables and suppose such a linear transformation made as shall convert the two numerators into 1 and x, so that the reducible integrals are

$$u = \int_{x_0 y_0}^{xy} \frac{dx}{y} \qquad \text{and} \qquad v = \int_{x_0 y_0}^{xy} \frac{x dx}{y}.$$

Let $a_1, a_2, a_3, a_4, a_5, a_6$ be the six zeros of $R_6(x) = 0$ and suppose them so numbered and the point x_0 so chosen that the a's may be joined to x_0 by lines which do not interfere and which succeed each other in the order one to six.

Let A_i denote the integral u taken from x_0, with the initial determination y_0 of y, along the line from x_0 to a_i, and let B_i denote the integral v along the same path. Then the following represent a system of normal periods.†

* It may be shown that those decompositions of $R_6(x_1 x_2)$ into four factors, two or more of which are of zero degree, arise from a reduction of degree k combined with a quadratic transformation of the elliptic integral.

† Briot, *Fonctions Abéliennes*, p. 64.

$$2A_1 - 2A_2,\ 2A_2 - 2A_3,\ 2A_1 - 2A_2 + 2A_3 - 2A_4,\ 2A_4 - 2A_5,$$

$$2B_1 - 2B_2,\ 2B_2 - 2B_3,\ 2B_1 - 2B_2 + 2B_3 - 2B_4,\ 2B_4 - 2B_5.$$

If we suppose that u and v are a pair of reducible integrals, then by the theorem of M. PICARD we may suppose this table identical with

$$1 \quad G \quad 0 \quad \frac{1}{D}$$

$$0 \quad \frac{1}{D} \quad 1 \quad G'.$$

Let $sn(u)$ be an elliptic function constructed with the periods

$$2K = \frac{1}{D}, \qquad 2iK' = G;$$

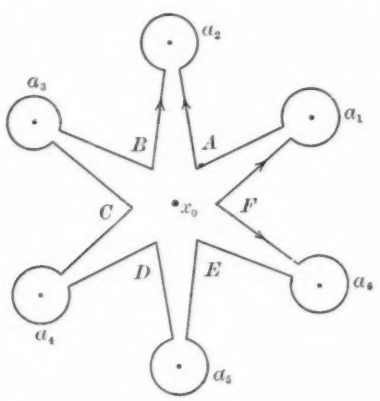
then according to M. PICARD, $sn(u)sn(2A_1 - u)$ is a rational function of x of degree D:

$$sn(u)sn(2A_1 - u) = \frac{f(x)}{F(x)}.$$

Expressing $sn(u)$ in terms of ϑ-functions, we have

$$sn(u) = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)},$$

$H(u)$ and $\Theta(u)$ being JACOBI's ϑ-functions and k a modulus corresponding to



the periods $1/D$ and G, the ϑ-functions being supposed constructed with the same periods. The following relations hold:

$$\Theta\left(u+\frac{1}{D}\right)=\Theta(u), \qquad \Theta(u+G)=-e^{-2i\pi D\left(u+\frac{G}{2}\right)}\Theta(u),$$

$$H\left(u+\frac{1}{D}\right)=-H(u), \qquad H(u+G)=-e^{-2i\pi D\left(u+\frac{G}{2}\right)}H(u).$$

We shall now find the sums of the zeros and infinities of the function

$$sn(u)\,sn(2A_1-u)=\frac{f(x)}{F(x)}.$$

To this end we suppose a star cut out of the x plane in the manner before described and illustrated in the figure.

The points $ABCDEF$ are supposed to be indefinitely near x_0.

If we suppose the integration carried out from a definite one of the inner points of the star, then u is a uniform function of x in the plane as cut. We shall choose the point F as initial point in integration.

Then if period multiples be ignored, to any value of x in the uncut plane correspond two values of the integral, viz., u and $2A_1-u$, and the sum of the zeros of $f(x)/F(x)$ is given according to the formula of CAUCHY by

$$\frac{1}{2\pi i}\int x d\log H(u)H(2A_1-u),$$

and the sum of the poles by

$$\frac{1}{2\pi i}\int x d\log \Theta(u)\Theta(2A_1-u),$$

the integrations being taken around the star in such a way as to leave the point ∞ on the left, or in the clockwise sense if the roots are arranged as in the figure.

The most convenient way to carry out these integrations* is to write

$$\int x d\log H(u)H(2A_1-u)=\int x\left[d\log H(2A_1-u)+d\log H(u)\right].$$

Reversing the order of the second integration, we get

$$\int x d\log H(2A_1-u)=-\int x d\log H(u),$$

the first integration being performed from F towards E and the second from F towards A.

If we let u_1 represent the value of the integral at a point on the opposite side

* PICARD, loc. cit., p. 50.

of the loop from where the value u is attained, as exhibited on loop 2, we may combine the two integrals into one:

$$\int x d \log \frac{H(2A_1 - u_1)}{H(u)} \equiv \int x d \log \frac{H(u')}{H(u)} \qquad (u' = 2A_1 - u_1).$$

This integral may be considered as the sum of six others taken along the six loops. We shall call these I_i $(i = 1, \cdots, 6)$.

To find these we observe that the difference $u' - u$ is constant along any loop, since u' and u are integrals with the same integrand function, as may be seen from counting the number of reversals of sign that have taken place in reaching the loop.

It suffices to calculate this difference at the beginning of the loop, that is, at corresponding points such as A and F.

We may conveniently denote by u_A, u_B, $\cdots$ the values of the integral at the points A, B, $\cdots$. Accordingly,

$$u'_A = 2A_1 - u_A = 2A_1 - 2A_1 = 0, \qquad u_F = 0,$$

and the difference along the first loop is $u'_A - u_F = 0$.

On loop 2 the difference is $u'_B - u_A = 2A_2 - 2A_1$, since

$$u'_B = 2A_1 - u_B = 2A_1 - 2A_1 + 2A_2, \qquad u_A = 2A_1.$$

In this way the successive differences may be found. They are here tabulated and expressed in terms of the periods $1/D$ and G.

Loop.	$u' - u$	$u' - u$
1	0	0
2	$2A_2 - 2A_1$	-1
3	$2A_2 - 2A_3 - (2A_1 - 2A_2)$	$G - 1$
4	$2A_2 - 2A_3$	G
5	$2A_2 - 2A_3 + 2A_4 - 2A_5$	$G + \frac{1}{D}$
6	$2A_2 - 2A_3 + 2A_4 - 2A_5$	$G + \frac{1}{D}$

Referring to the relations for the addition of periods to the argument of H, we find

$$I_1 = \int x d \log 1 = 0, \qquad I_2 = \int x d \log (\pm 1) = 0,$$

$$I_3 = \int x d \log \frac{H(u + G - 1)}{H(u)} = \int x d \log \left(\pm e^{-2\pi i D\left(u + \frac{G}{2}\right)} \right)$$

$$= -2\pi iD\int xdu = -2\pi iD\int dv,$$

$$I_4 = \int xd\log\frac{H(u+G)}{H(u)} = \int -2\pi iDx(-du) = 2\pi iD\int dv,$$

$$I_5 = \int xd\log\frac{H\left(u+G+\frac{1}{D}\right)}{Hu} = -2\pi iD\int dv,$$

$$I_6 = \int xd\log\frac{H\left(u+G+\frac{1}{D}\right)}{Hu} = 2\pi iD\int dv.$$

These integrals are supposed to be taken on their respective loops with the initial determination y_0 of y.

Referring to the table of periods of the integrals u and v we can express the integrals I in terms of B_i as follows:

$$I_1 = 0, \qquad I_2 = 0, \qquad I_3 = -2\pi iD\cdot 2B_3,$$

$$I_4 = 2\pi iD\cdot 2B_4, \qquad I_5 = -2\pi iD\cdot 2B_5, \qquad I_6 = 2\pi iD\cdot 2B_6.$$

Hence

$$\Sigma I_i = 2\pi iD(-2B_3 + 2B_4 - 2B_5 + 2B_6) = 2\pi iD(2B_1 - 2B_2) = 0,$$

since

$$2B_1 - 2B_2 + 2B_3 - 2B_4 + 2B_5 - 2B_6 = 0.$$

Since $\Sigma I_i = 0$, we have proved that

$$\int xd\log H(u)H(2A_1 - u) = 0$$

or the sum of the zeros of the rational function $f(z)/F(z)$ vanishes.

In essentially the same manner, we may prove that the sum of its infinities also vanishes.

If we write explicitly

$$\frac{f(x)}{F(x)} = \frac{a_0x^D + a_1x^{D-1} + \cdots + a_D}{b_0x^D + b_1x^{D-1} + \cdots + b_D},$$

then our results are simply that $a_1 = 0$, $b_1 = 0$.

We can now solve the problem which was proposed, viz., given a reducible hyperelliptic integral

$$\int \frac{P+Qx}{y}\,dx$$

and its reducing substitution $z = f(x)/F(x)$, to find the second integral

$$\int \frac{P' + Q'x}{y}\, dx$$

reducible by a transformation of the same degree.

Solution: Make the linear transformation

$$x = \frac{\alpha x' + \beta}{\gamma x' + \delta}$$

and so determine $\alpha, \beta, \gamma, \delta$ that the first reducible integral shall have a constant numerator and the reducing fraction $f(x)/F(x)$ shall be transformed into one wanting its terms of degree $D - 1$. Then the second reducible integral has a numerator proportional to x'.

We proceed to define this process in a way independent of linear transformation, which will at the same time show its uniqueness and thus complete the demonstration of its validity.

III. Application of the theory of invariants.

Let $f = a_x^n$ and $\phi = b_x^n$ be two binary forms of degree n. Then if the nth transvectant of f over ϕ or $(f, \phi)_n = (ab)^n$ vanishes, the forms f and ϕ are said to be apolar.

The apolarity condition extended is

$$a_0 b_n - n a_1 b_{n-1} + \frac{n(n-1)}{2} a_2 b_{n-2} - \cdots \pm a_n b_0 = 0.$$

Let $\lambda_1 f_1 + \mu_2 f_2$ be a pencil of binary forms of degree n denoted by (Σ). Then the forms apolar to (Σ) form an $n-2$-fold linear system $(\bar{\Sigma})$:

$$\mu_1 \phi_1 + \mu_2 \phi_2 + \cdots + \mu_{n-1} \phi_{n-1}.$$

Every form of either system is apolar to every form of the other.

In the pencil $\lambda_1 f_1 + \lambda_2 f_2$ occur $2n - 2$ double elements or linear forms $(x\delta)$ whose squares divide a form $\lambda_1 f_1 + \lambda_2 f_2$. These forms $(x\delta)$ occur $n-1$-fold in the system $(\bar{\Sigma})$ or $(x\delta)^{n-1}$ divides a form of $(\bar{\Sigma})$, and the complementary factor or quotient is unique. Denote this by $(x\delta')$. Then $(x\delta)^{n-1}(x\delta')$ is a form of $(\bar{\Sigma})$. Further, $(x\delta')$ is distinct from $(x\delta)$ unless $(x\delta)$ is a factor in every form $\lambda_1 f_1 + \lambda_2 f_2$, a possibility which we exclude.

If we make a linear transformation

$$x_1' = (x\delta'), \qquad x_2' = (x\delta),$$

then x_2' is a double element in (Σ) and $x_1' x_2'^{n-1}$ is a form of $(\bar{\Sigma})$.

When f_1 and f_2 are written explicitly in terms x_1', x_2',

$$f_1 \equiv a_0 x_1'^n + a_1 x_1'^{n-1} x_2' + \cdots, \qquad f_2 = b_0 x_1'^n + b_1 x_1'^{n-1} x_2' + \cdots,$$

the conditions that $x_1' x_2'^{n-1}$ is apolar to f_1 and f_2 are $a_1 = 0$, $b_1 = 0$. Conversely, if $a_1 = 0$, $b_1 = 0$, $x_1' x_2'^{n-1}$ will be apolar to f_1 and f_2.

Hence in order that the forms of a pencil may lack the terms next the highest it is necessary that the variables represent a double element in the pencil and its complementary element in that form of the conjugate system in which the double elements occur $n - 1$-fold.

Returning to the reduction problem, let us introduce homogeneous variables by putting $x = x_1/x_2$. Then the integrals and the reducing substitution become:

$$\int \frac{x_2(xdx)}{y}, \qquad \int \frac{x_1(xdx)}{y}, \qquad z = \frac{a_0 x_1^D + a_1 x_1^{D-1} x_2 + \cdots + a_D x_2^D}{b_0 x_1^D + b_1 x_1^{D-1} x_2 + \cdots + b_D x_2^D}.$$

Now we have proved that $a_1 = 0$, $b_1 = 0$.

Hence x_1 is the element conjugate to x_2^{n-1} in the system apolar to the forms

$$\lambda_1(a_0 x_1^D + \cdots + a_0 x_2^D) + \lambda_2(b_0 x_1^D \cdots + b_0 x_2^D).$$

Without using a special normal form let

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1 x_2)}}, \qquad \int \frac{(x\delta')(xdx)}{\sqrt{R_6(x_1 x_2)}},$$

$$\begin{cases} z_1 = f_1(x_1 x_2), \\ z_2 = f_2(x_1 x_2), \end{cases} \qquad \begin{cases} z_1 = f_1'(x_1 x_2), \\ z_2 = f_2'(x_1 x_2), \end{cases}$$

be two reducible integrals and the substitutions reducing them; then $(x\delta)^{D-1}(x\delta')$ is apolar to $\lambda_1 f_1 + \lambda_2 f_2$ and $(x\delta')^{D-1}(x\delta)$ is apolar to $\lambda_1 f_1' + \lambda_2 f_2'$. We have now proved the

THEOREM. *Given that*

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1 x_2)}}$$

is reducible by a transformation $z_1 = f_1(x_1 x_2)$, $z_2 = f_2(x_1 x_2)$ of order D. Determine $(x\delta')$ so that $(x\delta)^{D-1}(x\delta')$ belongs to the system apolar to $\lambda_1 f_1 + \lambda_2 f_2$. Then

$$\int \frac{(x\delta')(xdx)}{\sqrt{R_6(x_1 x_2)}}$$

is reducible by a transformation of order D.

The problem proposed is completely solved and the result expressed in a manner independent of linear transformation.

TORONTO, *July* 6, 1906.

ON THE DIFFERENTIAL INVARIANTS OF A PLANE*

BY

CHARLES NELSON HASKINS

The object of this note is the discussion of certain results recently obtained by FORSYTH † and RICCI concerning the differential invariants of plane curves.

§ 1. *Forsyth's† results.*

FORSYTH considers the differential form of class zero,

$$ds^2 = dx^2 + dy^2 = adu^2 + 2hdudv + bdv^2;$$

one or more curves $\phi(u, v) = \text{const.}$, their radii of curvature ρ, the normal derivatives $d\phi/du = C$, and the derivatives of ρ and C along the arc and along the normal.

He obtains the following results:

(1) "There are p^2 absolute invariants which involve derivatives of ϕ up to order p and are algebraically independent of one another."

(2) The quantities C and ρ, with their derivatives up to orders $p-1$ and $p-2$ respectively, form a system of p^2 geometrical magnitudes of orders not exceeding p, "being exactly the same in number as the aggregate of algebraically independent invariants up to the same order. Hence all the geometrical magnitudes are independent of one another, and therefore there are no relations among the geometrical magnitudes," (i. e.) "among the quantities C, ρ, and their derivatives common to all curves in the plane. This result differs from the geometry of curves upon a curved surface and from the geometry of surfaces in space."

§ 2. *Ricci's results and their consequences.*

RICCI ‡ has proved theorems which may be stated as follows:

(1) Quadratic differential forms of class zero have no Gaussian differential invariants.

(2) The differential parameters of quadratic differential forms of class zero are the simultaneous algebraic invariants of the given forms and the linear,

* Presented to the Society April 28, 1906. Received for publication August 25, 1906.

† FORSYTH, Rendiconti del Circolo Matematico di Palermo, vol. 21 (1906), pp. 115-125.

‡ RICCI, Annali di Matematica, ser. 2, vol. 14 (1885), p. 11.

quadratic, $\cdots$, p-ic forms whose coefficients are the first, second, $\cdots$, p-th covariant derivatives of ϕ.

It therefore follows from RICCI's results that:

(1) *There are*

$$3+2+3+\cdots+(p+1)-4=\frac{p^2+3p-2}{2}$$

absolute invariants which involve derivatives of ϕ of order up to p and are algebraically independent of one another.

(2) *There are*

$$R=p^2-\frac{p^2+3p-2}{2}=\frac{(p-1)(p-2)}{2}$$

relations among the p^2 geometrical magnitudes C, ρ, and their derivatives of orders not exceeding p.

§ 3. *Discussion of the results.*

The results of FORSYTH are obtained by consideration of the complete system of linear partial differential equations satisfied by the invariants in question — a method depending on the use of the infinitesimal transformations of the infinite group by which the form ds^2 is transformed.

RICCI's results are found by his method of the "Absolute Differential Calculus."

The two sets of results are contradictory except when $p=1$, or $p=2$. The reason for this is as follows. The number of invariants given by FORSYTH is obtained by subtracting from the number of independent variables in his complete system the number of equations comprising that system. Among these variables are the coefficients a, h, b, and their derivatives of various orders with respect to u and v. Were the form of class unity, that is, reducible to $dx^2+dy^2+dz^2$ but not to dx^2+dy^2, these variables would all be independent, and the number of invariants would be p^2, a result included in those given by ŻORAWSKI.* The form here under consideration is, however, of class zero; that is, reducible to dx^2+dy^2. Consequently the four-, five-, etc., index symbols of CHRISTOFFEL vanish identically, so that the derivatives of the coefficients a, h, b are no longer independent, but are bound by relations. These are

$$(12,12)=0,$$

$$(12,121)=0,$$

$$(12,122)=0,$$

* ŻORAWSKI, Acta Mathematica, vol. 16 (1892), p. 41.

$$(12, 1211) = 0,$$

$$(12, 1212) = 0,$$

$$(12, 1222) = 0,$$

etc.

In the case of invariants of order $p > 2$ there are

$$R = 1 + 2 + 3 + \cdots + (p-2) = \frac{(p-1)(p-2)}{2}$$

such relations, and hence only

$$p^2 - R = \frac{p^2 + 3p - 2}{2}$$

independent absolute invariants.

The geometrical magnitudes C, ρ, and their derivatives specified in (2), § 1, are absolute invariants, p^2 in number. This number is greater by R than the number of independent absolute invariants of order p. Hence there must exist R relations among these geometrical magnitudes. The lowest order for which $R \neq 0$ is $p = 3$; for which $R = 1$. Hence we should expect to find one relation among the geometrical magnitudes of the third order. It is easy to verify directly that the relation

$$\frac{1}{C}\frac{d^2C}{ds^2} = \frac{1}{\rho^2} - \frac{d}{dn}\left(\frac{1}{\rho}\right) + 2\left(\frac{1}{C}\frac{dC}{ds}\right)^2$$

is satisfied by the magnitudes in question. Moreover, FORSYTH† has already given, for curves on a surface, a relation among the geometrical magnitudes of the third order which reduces to this when the surface reduces to a plane.

MARION, MASS.,
August 22, 1906.

† FORSYTH, Philosophical Transactions, ser. A, vol 201 (1902), p. 398.

NOTES AND ERRATA: VOLUMES 6, 7

VOLUME 6

E. V. HUNTINGTON: *Note on the definitions ....*

P. 188. The proof of the commutative law for addition (theorem *A*7), here attributed to HILBERT, 1900, was given by H. HANKEL in 1867 (*Theorie der complexen Zahlensysteme*, p. 32).

H. MASCHKE: *Differential parameters of the first order.*

P. 69, § 1. The theorems on determinants developed here for the purpose of immediate application in this paper and in the paper "The Kronecker-Gaussian curvature of hyperspace" are not essentially new. I am indebted to Prof. W. H. METZLER for the following information:

P. 70, formula (1). The law which MUIR has called the law of extensible minors was known as early as 1867, for it was used by REIES in his "Beiträge zur Theorie der Determinanten" which was published in that year. MUIR proved it in 1881 (Transactions of the Royal Society, Edinburgh, vol. 30, part 1) and has since considered its applications (Proceedings Edinburgh Math. Soc., vol. 20, 1901–1902). Formula (1) is the result of applying this law to a theorem given by SYLVESTER (Phil. Mag., ser. 4, vol. 1) in 1851.

P. 71, formula (3). This theorem is the extensional (the result of applying the law of extensible minors) of a theorem given by PICQUET in 1878 (Journal de l'École Polyt., cah. 45, t. 28).